HITCHIN FIBRATIONS ON MODULI OF IRREGULAR HIGGS BUNDLES AND MOTIVIC WALL-CROSSING

PÉTER IVANICS, ANDRÁS STIPSICZ, AND SZILÁRD SZABÓ

ABSTRACT. In this paper we give a complete description of the Hitchin fibration on all 2-dimensional moduli spaces of rank-2 irregular Higgs bundles with two poles on the projective line. We describe the dependence of the singular fibers of the fibration on the eigenvalues of the Higgs fields, and describe the corresponding motivic wall-crossing phenomenon in the parameter space of parabolic weights.

1. INTRODUCTION

Moduli spaces of Higgs bundles with irregular singularities on Kähler manifolds have been extensively investigated over the last few decades from a variety of perspectives. One salient feauture of these spaces is the existence of a proper map to an affine space called the Hitchin fibration [11]. In mirror symmetric considerations, the singular fibers of this fibration play a major role, see relevant remarks in Subsection 1.2.

In this paper we study certain rank-2 irregular Higgs bundles $(\mathcal{E}, \theta)$ defined over $\mathbb{C}P^1$, where $\mathcal{E}$ is a rank-2 vector bundle and θ is a meromorphic section of $\mathcal{E}nd(\mathcal{E}) \otimes K$ called the Higgs field. We set $\deg(\mathcal{E}) = d$. We will limit ourselves to the case where θ has two poles q_1 and q_2, and the sum of the order of the poles is 4. The order of the poles are both 2 in the first subcase, and are 3 and 1 in the second subcase, hence (in the respective cases) θ is a holomorphic homomorphism

$$\theta : \mathcal{E} \to \mathcal{E} \otimes K(D) \tag{1}$$

where D is either $2 \cdot \{q_1\} + 2 \cdot \{q_2\}$ or $3 \cdot \{q_1\} + \{q_2\}$. Up to an isomorphism of $\mathbb{C}P^1$ we may fix the points $q_1 = [0 : 1]$ and $q_2 = [1 : 0]$.

We pick parameters $\alpha_i^j \in [0, 1)$ for $j \in \{1, 2\}$ and $i \in \{+, -\}$ and for simplicity we write $\vec{\alpha} = (\alpha_+^1, \alpha_-^1, \alpha_+^2, \alpha_-^2)$. The moduli spaces of interest to us (parameterizing $\vec{\alpha}$-(semi-)stable irregular Higgs bundles of rank 2 over $\mathbb{C}P^1$ with two poles and fixed polar part) will be denoted $\mathcal{M}^{(s)s}(\vec{\alpha})$. By definition $\mathcal{M}^s(\vec{\alpha}) \subset \mathcal{M}^{ss}(\vec{\alpha})$, and $\mathcal{M}^{ss}(\vec{\alpha}) \setminus \mathcal{M}^s(\vec{\alpha})$ is finite. These spaces are equipped with a morphism

$$h : \mathcal{M}^{(s)s}(\vec{\alpha}) \to B, \tag{2}$$

called the ***irregular Hitchin map***, where B is an affine line over $\mathbb{C}$. This map is a straightforward generalization of the Hitchin map on moduli spaces of holomorphic Higgs bundles [11]. For details on the irregular Hitchin map, see Subsection 3.3. The moduli spaces depend on some further parameters that we will tag (S), (N), (s) and (n). In this paper we give a complete description of the singular fibers of h depending on the parameters, under the following condition.

Assumption 1.1. *At least one (equivalently, the generic) fiber of h is a smooth elliptic curve.*

The irregular Hitchin map h extends to a map $\overline{h}: \mathcal{M}^{ss} \cup E_\infty \to \mathbb{C}P^1$, where E_∞ is a complex curve, which is the fiber at infinity of the fibration $\overline{h}$. It turns out that $\overline{h}$ is

Key words and phrases. irregular Higgs bundles, Hitchin fibration, wall crossing, elliptic fibrations.
Corresponding author: Szilárd Szabó.

generically an elliptic fibration on the complex surface $\mathcal{M}^{ss} \cup E_\infty$, which, on the other hand, is a (Zariski) open subset of a rational elliptic surface (which surface is diffeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$). The singular fibers of an elliptic fibration have been classified by Kodaira [15], and the relevant singular fibers will be recalled in Section 4. The *combinatorial type* of an elliptic fibration is the list of the singular fibers (with multiplicity) arising in the particular fibration. In [16, 19, 22] the complete list of all possible combinatorial types of elliptic fibrations on a rational elliptic surface have been determined.

This classification turns out to be useful in understanding irregular Hitchin fibrations. In the following we will examine various cases of Equation (2): we deal with the two cases when $D = 2q_1 + 2q_2$ or $D = 3q_1 + q_2$ in Equation (1), and in each case we have to separate further subcases depending on behaviour of the Higgs field at the poles — indeed, we will distinguish regular semi-simple (denoted by (S) for the pole at q_1 and by (s) at q_2) and nilpotent cases (denoted by (N) and (n), respectively). Indeed, for $D = 2q_1 + 2q_2$ we will have three subcases to distinguish (listed in Subsection 2.1), while for $D = 3q_1 + q_2$ there are four distinct cases (listed in Subsection 2.2). In each case the map h depends on the complex parameters defining the moduli space; these cases are denoted $(S), (N), (s)$ and (n) in Section 2.

Before giving the precise (and somewhat tedious) forms of our results, here we just state the main principle. For the exact formulae and the possible combinatorial types of the seven cases see the expanded versions of the Main Theorem in Section 2.

Main Theorem 1.2. *In each case there is a precise formula in terms of the complex parameters of the polar part of the Higgs field which determines the class in the Grothendieck ring of varieties of all individual singular fibers of the Hitchin fibration.*

Notice that some of the Hitchin fibers we find do not belong to Kodaira's list because the Hitchin fibers may be noncompact. We explain this phenomenon in detail at the beginning of Section 2. (This phenomenon has been also discussed in the Painlevé VI case [13, Proposition 2.9].)

Theorem 1.2 is different from the authors' results [14] in the case of a single irregular singularity; in fact the latter case arises as a degeneration of the setup of the present paper.

The statements and arguments in the paper seem to be rather repetitive. Although the driving ideas in the cases are very similar, the actual shapes of the computations (and henceforth the statements themselves) are quite different. Indeed, a wall crossing phenomenon (to be detailed in the next Subsection) arises in three of the seven cases. The key idea in each proof is that we determine the number of roots of certain polynomials. In Theorems 2.1 and 2.4 use a conversion to symmetric polynomials and other special polynomials to find the singular fibers of the fibration, while the proof of Theorem 2.3, on the other hand, is rather simple. The proofs of Theorems 2.5 and 2.7 involve the essential use of the blow-up procedure. For the sake of completeness we therefore decided to give full arguments rather than just sketching the proofs in the individual subcases.

1.1. **Wall-crossing.** In this Subsection we highlight a feature of the technical results of the paper. Namely, in the detailed versions of Theorem 1.2 we determine the diffeomorphism class (or, in some cases the class in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$, cf. Subsection 3.4) of the fibers of the irregular Hitchin map. Now, in the assertions where only the class of the fiber in $K_0(\mathrm{Var}_{\mathbb{C}})$ is specified, we could be more precise by attaching integer indices (δ_+, δ_-) to the components of the given fiber corresponding to the bidegree imposed on the torsion-free sheaves parametrized by the given class. For the notion of bidegree we refer to (86). In Subsections 8.6, 8.7, 9.1 and 9.2 we do include the bidegree in the notation as an index of the components of the Hitchin fiber. The notions of generic and special parabolic weight will be provided in Definition 8.14. It turns out that all possible parabolic weights constitute a real vector space of dimension one and special weigths form $\mathbb{Z} \subset \mathbb{R}$. We call this copy of $\mathbb{Z} \subset \mathbb{R}$ the set of ***walls***. In Sections 8 and 9 we will prove the following simple wall-crossing result concerning cases (1), (5) and (6) of Theorem 2.1, cases (1), (2) and (3) of Theorem 2.4, and cases (1) and (3) of Theorem 2.5.

Theorem 1.3. *The class of the singular fiber in* $K_0(\mathrm{Var}_{\mathbb{C}})$ *is the same on both sides of a wall, but the index of the given classes changes from* (δ_+, δ_-) *to* $(\delta_+ \pm 1, \delta_- \mp 1)$.

From a physical point of view, this wall-crossing phenomenon was studied in [4, Subsection 9.4.6].

1.2. **Mirror symmetry and Langlands duality.** Let us now comment on one consequence of our results. Namely, Lemma 10.1 of Section 10 may be reformulated as saying that in the degenerate cases one of the Hitchin fibers is a certain Jacobian within a compactified Jacobian of a singular curve, and the compactifying point corresponds to a Higgs bundle with the required eigenvalues but vanishing nilpotent part. Hence, as may be expected, the completion of these moduli spaces arises by allowing for more special residue conditions with the same characteristic polynomial. This phenomenon may be interpreted as an Uhlenbeck-type compactification result for moduli spaces of irregular Higgs bundles. Indeed, as we will see in Lemma 4.5, in the degenerate cases one singular spectral curve has one component which is an exceptional divisor in the blow-up of the Hirzebruch surface. Now, under suitable degree conditions the direct image with respect to the ruling morphism p of sheaves on such a special curve gives rise to a non-locally free sheaf on the base curve, i.e. a sheaf with one fiber of dimension higher than 2 — an analogue of infinitely concentrated (or Dirac) instantons from gauge theory in the context of irregular Higgs bundles.

It is known [5, 10] that Hitchin moduli spaces $\mathcal{M}_G(C)$ and $\mathcal{M}_{^LG}(C)$ on a given curve C corresponding to Langlands dual groups $G, ^L G$ are mirror partners in the sense of Strominger–Yau–Zaslow [23]. Moreover, it is expected from mirror symmetry considerations of Gukov [9, 12] that BAA-branes (flat bundles over Lagrangian subvarieties) on $\mathcal{M}_G(C)$ should be mirror to BBB-branes (hyperholomorphic sheaves) on $\mathcal{M}_{^LG}(C)$. In [12, Section 7], Hitchin proposes a candidate for such a mirror dual pair in the case $G = \mathrm{Gl}(2m, \mathbb{C})$. The role of the BAA-brane in this setup is played by the trivial bundle over the character variety for the real form $G^r = U(m, m)$, with corresponding BBB-brane a certain holomorphic vector bundle over the moduli space of $\mathrm{Sp}(2m, \mathbb{C})$ Higgs bundles. Essentially, Hitchin proves that away from the discriminant locus (i.e. for Higgs bundles with smooth spectral curve) the kernel of an even exterior power of a certain Dirac-operator and the moduli space of $U(m, m)$-Higgs bundles with given characteristic classes are in a Fourier–Mukai type of duality. The interpretation of this relationship as a Fourier–Mukai transform breaks down over the singular fibers (which are no longer tori). We hope that our results, providing a complete understanding of the singular fibers of the irregular version of the Hitchin map, will be of use in order to verify a similar phenomenon for the spaces we consider.

As for an application of our results in another direction, in [24] the third named author computes the perverse filtration on the cohomology of the 2-dimensional moduli spaces of irregular parabolic Higgs bundles on the projective line, and compares them to the mixed Hodge structure on the corresponding wild character variety.

The paper is composed as follows. In Section 2 we provide the full statements of Main Theorem 1.2 in the various cases. In Section 3 we collect some (mostly standard) material about irregular Higgs bundles, their moduli spaces, the analog of Hitchin's fibration in this setup and the Grothendieck ring of varieties. In Section 4 we discuss some general properties of elliptic fibrations on rational elliptic surfaces. In Sections 5 and 6 we carry out a complete analysis of the singular fibers of the elliptic fibrations obtained from an elliptic pencil on a Hirzebruch surface, in terms of the parameters specifying their base locus. Finally, in Sections 7, 8, 9 and 10 we determine the families of torsion-free sheaves supported on the singular curves giving rise to $\vec{\alpha}$-(semi-)stable irregular Higgs bundles.

Acknowledgements: A. S. was supported by ERC Advanced Grant LDTBud and by NKFIH 112735. A. S. and Sz. Sz. were supported by the *Lendület* grant Low Dimensional Topology of the Hungarian Academy of Sciences. P. I. and Sz. Sz. were partially supported by NKFIH 120697. The authors are grateful to an anonymous referee for many insightful comments and suggestions.

2. THE PRECISE VERSIONS OF THE MAIN THEOREM

Now we turn to stating the precise versions of our resuts, which have been summarized in Main Theorem 1.2. For the sake of simplicity, in this section and in Section 10 we loosen standard terminology as follows: by elliptic fibration we mean a morphism $X \to C$ from a (possibly non-compact) surface X to a compact curve C if the generic fiber is a compact smooth elliptic curve. Irregular Hitchin fibrations on the moduli spaces of Higgs bundles under consideration are biholomorphic to the complement of one singular fiber in an elliptic fibration in this more general sense. The fundamental reason that the fibration is elliptic only in this broader sense is that in case the orbit of the residue of the Higgs field at the logarithmic point is non-semisimple, a sequence of Higgs bundles with given characteristic polynomial and with residue in the non-semisimple orbit may converge to a Higgs bundle with the same characteristic polynomial and residue in the closure of the given orbit (rather than the orbit itself). The geometric manifestation of this phenomenon is the existence of some irreducible components of fibers in elliptic fibrations mapping to a point under the ruling of the Hirzebruch surface. In these (so-called degenerate) cases, one needs a finer analysis of the possible spectral sheaves giving rise to parabolically stable Higgs bundles. This analysis will be carried out in Section 10, and will show the existence of non-compact fibers (even though the spectral curves themselves are compact). It follows that the elliptic fibrations that we discuss throughout the paper are honest elliptic fibrations in the usual sense (as they are obtained from pencils of spectral curves), except for this section and Section 10. We chose to keep the usual terminology for surfaces with non-compact fibers because non-compactness only appears at the last step, where a simple comparison of classes in Grothendieck ring makes it obvious which fibers are not compact.

2.1. **Statement of results in the Painlevé III cases.**

Consider first the case when the order of the poles of θ is 2 at both points. We distinguish three subcases, according to whether the polar part of the Higgs field is semisimple (referred to as (S) or (s)) or has nonvanishing nilpotent part (referred to as (N) or (n)) near q_1, q_2. Via nonabelian Hodge theory, the corresponding meromorphic connections of these subcases give rise to

(1) $PIII(D6)$ when both polar parts are semisimple;
(2) $PIII(D7)$ when exactly one polar part is semisimple and the other one has nonvanishing nilpotent part;
(3) $PIII(D8)$ when both polar parts have nonvanishing nilpotent part.

To define the moduli spaces of irregular Higgs bundles in these cases, we need to fix some parameters. Namely, depending on the cases, we need to fix sets of parameters of the form $(Ss), (Sn), (Ns), (Nn)$ where the letters S, N, s, n refer to the following sets of natural complex parameters

$$
\begin{array}{rl}
(S) & a_+, a_-, \lambda_+, \lambda_- \\
(N) & a_{-4}, a_{-3}, a_{-2} \\
(s) & b_+, b_-, \mu_+, \mu_- \\
(n) & b_{-4}, b_{-3}, b_{-2}.
\end{array}
$$

The parameters appearing in the above lists have geometric meaning: basically they encode the base locus of an elliptic pencil on the Hirzebruch surface $\mathbb{F}_2$, see (20), (21), (22), (23). For example, a_+, a_- (and similarly b_+, b_-) determine the locations of the base points on the fibers of the Hirzebruch surface, see also Figures 4 and 5. In order to state our results, it will be useful to consider some polynomial expressions M, L, A and B of these parameters:

$$A = a_- - a_+, \quad B = b_- - b_+, \quad L = \lambda_- - \lambda_+, \quad M = \mu_- - \mu_+. \tag{3}$$

In the case (S) (or (s)), if the condition $A \neq 0$ (respectively, $B \neq 0$) holds, we call the semisimple polar part ***regular***. Geometrically, this amounts to requiring that there are two distinct base points on the corresponding fiber.

Moreover, we refer to Definition 8.14 for the notion of generic and singular parabolic weights. In the statements of our results we need one more (abstract) concept: By the class of a variety we mean its class in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of varieties, see Subsection 3.4. In particular, $\mathbf{L}$ stands for the class of the affine line and $\mathbf{1}$ for that of a point; as a consequence $\mathbf{L}+\mathbf{1}$ and $\mathbf{L}-\mathbf{1}$ denote the classes of $\mathbb{C}P^1$ and $\mathbb{C}^\times$, respectively. Finally, for notations and (standard) conventions regarding singular elliptic fibers (from Kodaira's list [15]), see Section 4.

In the next theorem $\Delta = -256A^3B^3 + 192A^2B^2LM - 3AB\left(9L^4 - 2L^2M^2 + 9M^4\right) + 4L^3M^3$ (a certain discriminant naturally associated to a degree-4 polynomial specified by the problem).

Theorem 2.1 (PIII(D6))**.** *Assume that the polar part of the Higgs field is of order* 2 *and regular semisimple both near* q_1 *and near* q_2*, that is, we are in case* (Ss)*. Then the irregular Hitchin fibration* h *on* $\mathcal{M}^{ss}(\vec{\alpha})$ *is biregular to the complement of the fiber at infinity which is of type* I_2^* *(equivalently* $\tilde{D}_6$*) in an elliptic fibration of the rational elliptic surface such that the set of the other singular fibers is:*

1. *if* $\Delta = 0$ *and* $L^2 = M^2 \neq 0$ *then an* I_1 *curve and*
   (a) *for generic weights a further type* III *curve,*
   (b) *for special weights a fiber in the class* $\mathbf{L}+\mathbf{1}$*, with the class of the corresponding fiber of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L}$*;*
2. *if* $\Delta = 0$*,* $L^2 = -M^2 \neq 0$ *and* $M^3 = 8ABL$*, then two type* II *fibers;*
3. *if* $\Delta = 0$*,* $L^2 = -M^2 \neq 0$ *and* $M^3 \neq 8ABL$*, then a type* II *and two* I_1 *fibers;*
4. *if* $\Delta = 0$ *and* $L^2 \neq \pm M^2$*, then a type* II *and two* I_1 *fibers again;*
5. *if* $\Delta \neq 0$ *and* $L = M = 0$ *then*
   (a) *for generic weights two type* I_2 *fibers,*
   (b) *for special weights two fibers in the class* $\mathbf{L}$*, with the classes of the corresponding fibers of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L}-\mathbf{1}$*;*
6. *if* $\Delta \neq 0$ *and* $L^2 = M^2 \neq 0$ *then two* I_1 *fibers and*
   (a) *for generic weights a further type* I_2 *fiber,*
   (b) *for special weights a fiber in the class* $\mathbf{L}$*, with the class of the corresponding fiber of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L}-\mathbf{1}$*;*
7. *if* $\Delta \neq 0$ *and* $L^2 \neq M^2$*, then four type* I_1 *fibers.*

We note that $\Delta = 0$ *and* $L = M = 0$ *implies* $A = 0$ *or* $B = 0$*, hence this case is not in* (Ss)*, therefore the above items cover all cases.*

In the next theorem we use the discriminant $\Delta = 4A^3b_{-3}\left(2L^3 - 27Ab_{-3}\right)$ in the case (Sn) and $\Delta = 4B^3a_{-3}\left(2M^3 - 27Ba_{-3}\right)$ in the case (Ns).

Theorem 2.2 (PIII(D7))**.** *Assume that the polar part of the Higgs field is of order* 2 *and regular semisimple near* q_1 *and of order* 2 *and non-semisimple near* q_2 *(or vice versa), i.e we are in case* (Sn) *(or in* (Ns)*, respectively). Then the irregular Hitchin fibration* h *on* $\mathcal{M}^{ss}(\vec{\alpha})$ *is biregular to the complement of the fiber at infinity which is of type* I_3^* *(equivalently* $\tilde{D}_7$*) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

1. *if* $\Delta = 0$*, then a type* II *and an* I_1 *fibers;*
2. *if* $\Delta \neq 0$*, then three type* I_1 *fibers.*

Theorem 2.3 (PIII(D8)). *Assume that the polar part of the Higgs field is of order* 2 *and non-semisimple both near* q_1 *and near* q_2*, i.e. we are in* (Nn)*. Then the irregular Hitchin fibration* h *on* $\mathcal{M}^{ss}(\vec{\alpha})$ *is biregular to the complement of the fiber at infinity which is of type* I_4^* *(equivalently* $\tilde{D}_8$*) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $a_{-3}b_{-3} \neq 0$*, then two type* I_1 *fibers.*

We note that if $a_{-3} = 0$ *or* $b_{-3} = 0$ *then the fibration is not elliptic.*

2.2. **Statement of results in the Painlevé II and IV cases.** Next we turn to the cases when the orders of the poles of θ are 3 at q_1 and 1 at q_2. Just as before, we distinguish several subcases, once again based on semisimplicity. In this case, we have four subcases, and the corresponding results read as follows. The natural parameters for the moduli of Higgs bundles are again of the form $(Ss), (Sn), (Ns), (Nn)$, this time the letters S, N, s, n referring to sets of natural complex parameters

$$\begin{array}{rl} (S) & a_+, a_-, b_+, b_-, \lambda_+, \lambda_- \\ (N) & b_{-6}, b_{-5}, b_{-4}, b_{-3}, b_{-2} \\ (s) & \mu_+, \mu_- \\ (n) & b_{-1} \end{array}$$

of geometric meaning detailed in (51), (52), (53), (54), see also Figures 6, 7 and 8. Once again, we derive new symbols M, L, A and B out of these natural parameters as follows:
(4)

$$A = a_- - a_+, \quad B = b_- - b_+, \quad L = \lambda_- - \lambda_+, \quad M = \mu_- - \mu_+, \quad Q = 8b_{-5}, \quad R = b_{-4}^2 + 4b_{-3}.$$

Again, the semisimple polar parts are called *regular* if $A \neq 0$ (respectively, $M \neq 0$).

In the next theorem the appropriate discriminant Δ is equal to

$$48A^4\left(L^2 + 3M^2\right)^2 + 64A^3B^2L\left(L^2 - 9M^2\right) + 24A^2B^4\left(L^2 + 3M^2\right) - B^8.$$

Theorem 2.4 (PIV). *Assume that the polar part of the Higgs field is of order* 3 *and regular semisimple near* q_1 *and of order* 1 *and regular semisimple near* q_2*, i.e., we consider the case (Ss). Then the irregular Hitchin fibration* h *on* $\mathcal{M}^{ss}(\vec{\alpha})$ *is biregular to the complement of the fiber at infinity which is of type* $\widetilde{E}_6$ *in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $L = \pm M$ *and* $B^2 = \pm 4AM$ *(consequently* $\Delta = 0$*) then an* I_1 *fiber and*
   (a) for generic weights a type III *fiber,*
   (b) for special weights a fiber in the class $\mathbf{L} + \mathbf{1}$*, with the class of the corresponding fiber of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L}$*;*

(2) if $L = \pm M$ *and* $B^2 = \mp 12AM$ *(consequently* $\Delta = 0$*) then a type* II *fiber and*
   (a) for generic weights an I_2 *fiber,*
   (b) for special weights a fiber in the class $\mathbf{L}$*, with the class of the corresponding fiber of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L} - \mathbf{1}$*;*

(3) if $L = \pm M$*,* $B^2 \neq \pm 4AM$ *and* $B^2 \neq \mp 12AM$ *(consequently* $\Delta \neq 0$*) then two* I_1 *fibers and*
   (a) for generic weights an I_2 *fiber,*
   (b) for special weights a fiber in the class $\mathbf{L}$*, with the class of the corresponding fiber of* $\mathcal{M}^s(\vec{\alpha})$ *given by* $\mathbf{L} - \mathbf{1}$*;*

(4) if $L \neq \pm M$ *and* $\Delta \neq 0$*, four type* I_1 *fibers.*

(5) if $L \neq \pm M$*,* $\Delta = 0$ *and* $B = 0$*, then two type* II *fibers;*

(6) if $L \neq \pm M$*,* $\Delta = 0$ *and* $B \neq 0$*, then a type* II *and two* I_1 *fibers;*

In the next theorem $\Delta = 4A^2\left(B^2 - 6AL\right)$.

Theorem 2.5 (Degenerate PIV). *Assume that the polar part of the Higgs field is of order* 3 *and regular semisimple near* q_1 *and of order* 1 *and non-semisimple near* q_2*, that is, we* *are in case (Sn). Then the irregular Hitchin fibration h on $\mathcal{M}^{ss}(\vec{\alpha})$ is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_6$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $\Delta = 0$ and $L = 0$ then
  (a) for generic weights a fiber in the class $2\mathbf{L}$,
  (b) for special weights a fiber in the class $\mathbf{L}+\mathbf{1}$, with the class of the corresponding fiber of $\mathcal{M}^s(\vec{\alpha})$ given by $\mathbf{L}$;

(2) if $\Delta = 0$ and $L \neq 0$, then a type II fiber and a fiber in the class $\mathbf{L} - \mathbf{1}$;

(3) if $\Delta \neq 0$ and $L = 0$ then an I_1 fiber and
  (a) for generic weights a fiber in the class $2\mathbf{L} - \mathbf{1}$,
  (b) for special weights a fiber in the class $\mathbf{L}+\mathbf{1}$, with the class of the corresponding fiber of $\mathcal{M}^s(\vec{\alpha})$ given by $\mathbf{L}$;

(4) if $B^2 = -2AL$ and $L \neq 0$ (consequently $\Delta \neq 0$) then an I_1 fiber and a fiber in the class $\mathbf{L}$;

(5) if $\Delta \neq 0$ and $L \neq 0$ and $B^2 \neq -2AL$ then two I_1 fibers and a fiber in the class $\mathbf{L} - \mathbf{1}$.

In the next theorem $\Delta = M^2\left(27M^2Q^2 - 4R^3\right)$.

Theorem 2.6 (PII)**.** *Assume that the polar part of the Higgs field is of order 3 and non-semisimple near q_1 and of order 1 and regular semisimple near q_2, i.e. we are in case (Ns). Then the irregular Hitchin fibration h on $\mathcal{M}^{ss}(\vec{\alpha})$ is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $\Delta = 0$ and $Q \neq 0$, then a type II and an I_1 fiber;

(2) if $\Delta \neq 0$ and $Q \neq 0$, then three type I_1 fibers.

We note that if $Q = 0$ (equivalently, $b_{-5} = 0$), then the fibration is not elliptic.

Theorem 2.7 (Degenerate PII)**.** *Assume that the polar part of the Higgs field is of order 3 and non-semisimple near q_1 and of order 1 and non-semisimple near q_2, i.e., we are in (Nn). Then the irregular Hitchin fibration h on $\mathcal{M}^{ss}(\vec{\alpha})$ is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $R = 0$ and $Q \neq 0$, then a fiber in the class $\mathbf{L}$;

(2) if $R \neq 0$ and $Q \neq 0$ then an I_1 fiber and a fiber in the class $\mathbf{L} - \mathbf{1}$;

We note that if $Q = 0$ (equivalently, $b_{-5} = 0$), then the fibration is not elliptic.

3. Preparatory material

3.1. **Irregular Higgs bundles of rank 2 on curves.** Let C be a smooth projective curve over $\mathbb{C}$ and D an effective Weil divisor over C (possibly non-reduced). Throughout the main body of this paper we will be interested in the case $C = \mathbb{C}P^1$ and

$$D = 2 \cdot \{q_1\} + 2 \cdot \{q_2\} \tag{2,2}$$

or

$$D = 3 \cdot \{q_1\} + \{q_2\} \tag{3,1}$$

for some distinct points $q_1, q_2 \in \mathbb{C}P^1$.

Definition 3.1. *A rank-2 **irregular Higgs bundle** is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a rank-2 vector bundle over C and*

$$\theta \in H^0(C, \mathcal{E}nd(\mathcal{E}) \otimes K_C(D)).$$

For the local forms of the Higgs fields that we will consider, see (20), (22), (51) and (53) (regular semisimple case) and (21), (23), (52) and (54) (non-semisimple case).

Definition 3.2. *A **compatible quasi-parabolic structure** on an irregular Higgs bundle $(\mathcal{E}, \theta)$ at q_j is the choice of a generalized eigenspace of the leading order term of θ at q_j with respect to some local coordinate. A **compatible parabolic structure** on $(\mathcal{E}, \theta)$ at q_j is a compatible quasi-parabolic structure endowed with a real number $\alpha_i^j \in [0, 1)$ (the **parabolic weight**) attached to every generalized eigenspace of the leading order term of θ at q_j.*

A compatible quasi-parabolic structure for a Higgs bundle with non-semisimple singular part at q_j is redundant, whereas in the regular semi-simple case it amounts to the choice of one of the two eigendirections of its expansion. A compatible parabolic structure for a Higgs bundle with non-semisimple singular part at q_j is just the choice of a parameter $\alpha^j \in [0, 1)$, whereas in the regular semi-simple case it amounts to the choice of a parabolic weight $\alpha_i^j \in [0, 1)$ for each eigenvalue of its most singular term, where $i \in \{+, -\}$. For coherence of notations, even in the non-semisimple case we will use the notations α_+^j and α_-^j and set

$$\alpha_+^j = \alpha_-^j = \alpha^j. \tag{5}$$

This convention reflects the fact that α^j has multiplicity 2.

Definition 3.3. *The **parabolic degree** of an irregular Higgs bundle $(\mathcal{E}, \theta)$ endowed with a parabolic structure at both q_1, q_2 is*

$$\deg_{\vec{\alpha}}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=1}^{2} \sum_{i \in \{+,-\}} \alpha_i^j,$$

*where $\deg(\mathcal{E}) = \langle c_1(\mathcal{E}), [C] \rangle$. The **parabolic slope** of $(\mathcal{E}, \theta)$ is*

$$\mu_{\vec{\alpha}}(\mathcal{E}) = \frac{\deg_{\vec{\alpha}}(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} = \frac{\deg_{\vec{\alpha}}(\mathcal{E})}{2}.$$

If $\deg_{\vec{\alpha}}(\mathcal{E}) = 0$, then non-abelian Hodge theory establishes a diffeomorphism between irregular Dolbeault and de Rham moduli spaces [2]. Moreover, the combinatorics of the stability condition and the resulting geometry of the singular Hitchin fibers would be very similar if we fixed the parabolic degree to be equal to some other constant. Therefore we make the following

Assumption 3.4. *We will suppose $\deg_{\vec{\alpha}}(\mathcal{E}) = 0$.*

Definition 3.5. *A rank-1 **irregular Higgs subbundle** of an irregular Higgs bundle $(\mathcal{E}, \theta)$ is a couple $(\mathcal{F}, \theta_{\mathcal{F}})$ where $\mathcal{F} \subset \mathcal{E}$ is a rank-1 subbundle such that θ restricts to*

$$\theta_{\mathcal{F}} \in H^0(C, \mathcal{E}nd(\mathcal{F}) \otimes K_C(D)).$$

In the rank-2 case, non-trivial Higgs subbundles are exactly rank 1 Higgs subbundles, hence from now on we only deal with rank-1 subbundles. It is easy to see that if $(\mathcal{F}, \theta_{\mathcal{F}})$ is an irregular Higgs subbundle then for both $j \in \{1, 2\}$ the fiber $\mathcal{F}_{q_j}$ of $\mathcal{F}$ at q_j must be a subspace of one of the generalized eigenspaces of the leading order term of θ at q_j. Notice that the fiber $\mathcal{F}_{q_j}$ has no non-trivial filtrations. These observations show that the following definition makes sense.

Definition 3.6. *The **induced parabolic structure** on an irregular Higgs subbundle $(\mathcal{F}, \theta_{\mathcal{F}})$ is the choice of the parabolic weight*

$$\alpha^j(\mathcal{F}) = \alpha_i^j$$

*where α_i^j is the parabolic weight of $\mathcal{E}$ at q_j corresponding to the generalized eigenspace containing $\mathcal{F}_{q_j}$. The **parabolic degree** of $(\mathcal{F}, \theta_{\mathcal{F}})$ is*

$$\deg_{\vec{\alpha}}(\mathcal{F}) = \deg(\mathcal{F}) + \sum_{j=1}^{2} \alpha^j(\mathcal{F}).$$

*The **parabolic slope** of $(\mathcal{F}, \theta_{\mathcal{F}})$ is*

$$\mu_{\vec{\alpha}}(\mathcal{F}) = \deg_{\vec{\alpha}}(\mathcal{F}).$$

For an irregular Higgs subbundle $(\mathcal{F}, \theta|_{\mathcal{F}})$ of $(\mathcal{E}, \theta)$, θ induces a morphism on the quotient vector bundle

$$\mathcal{Q} = \mathcal{E}/\mathcal{F};$$

we denote the resulting irregular Higgs field by

$$\bar{\theta} : \mathcal{Q} \to \mathcal{Q} \otimes K_C(D).$$

Definition 3.7. *A **quotient irregular Higgs bundle** of $(\mathcal{E}, \theta)$ is the irregular Higgs bundle $(\mathcal{Q}, \bar{\theta})$ obtained as above for some irregular Higgs subbundle $(\mathcal{F}, \theta_{\mathcal{F}})$ of $(\mathcal{E}, \theta)$. The **induced parabolic structure** on a quotient irregular Higgs bundle $(\mathcal{Q}, \bar{\theta})$ is defined by the parabolic weight $\alpha^j(\mathcal{Q})$ such that*

$$\{\alpha^j(\mathcal{F}), \alpha^j(\mathcal{Q})\} = \{\alpha_+^j, \alpha_-^j\}.$$

*The **parabolic slope and degree** of a quotient irregular Higgs bundle $(\mathcal{Q}, \bar{\theta})$ are defined as*

$$\mu_{\vec{\alpha}}(\mathcal{Q}) = \deg_{\vec{\alpha}}(\mathcal{Q}) = \deg(\mathcal{Q}) + \sum_{j=1}^{2} \alpha^j(\mathcal{Q}).$$

Let $(\mathcal{F}, \theta|_{\mathcal{F}})$ be an irregular Higgs subbundle of $(\mathcal{E}, \theta)$ and $(\mathcal{Q}, \bar{\theta})$ be the corresponding quotient irregular Higgs bundle of $(\mathcal{E}, \theta)$. Then, by additivity of the degree we have

$$\deg_{\vec{\alpha}}(\mathcal{F}) + \deg_{\vec{\alpha}}(\mathcal{Q}) = \deg_{\vec{\alpha}}(\mathcal{E}).$$

Definition 3.8. *An irregular Higgs bundle $(\mathcal{E}, \theta)$ is*

- ***$\vec{\alpha}$-semi-stable** if for any non-trivial irregular Higgs subbundle $(\mathcal{F}, \theta_{\mathcal{F}})$ we have*
$$\deg_{\vec{\alpha}}(\mathcal{F}) \leq \frac{\deg_{\vec{\alpha}}(\mathcal{E})}{2};$$
equivalently, if for any non-trivial irregular quotient Higgs bundle $(\mathcal{Q}, \bar{\theta})$ we have
$$\deg_{\vec{\alpha}}(\mathcal{Q}) \geq \frac{\deg_{\vec{\alpha}}(\mathcal{E})}{2};$$
- ***$\vec{\alpha}$-stable** if the corresponding strict inequalities hold in the definition of $\vec{\alpha}$-semi-stability;*
- ***strictly $\vec{\alpha}$-semi-stable** if it is $\vec{\alpha}$-semi-stable but not $\vec{\alpha}$-stable;*
- ***$\vec{\alpha}$-polystable** if it is a direct sum of two rank-1 irregular Higgs bundles of the same parabolic slope as $(\mathcal{E}, \theta)$;*
- ***strictly $\vec{\alpha}$-polystable** if it is $\vec{\alpha}$-polystable but not $\vec{\alpha}$-stable.*

Because of Assumption 3.4, the $\vec{\alpha}$-semi-stability condition boils down to

$$\deg_{\vec{\alpha}}(\mathcal{F}) \leq 0$$

and

$$\deg_{\vec{\alpha}}(\mathcal{Q}) \geq 0,$$

and similarly for $\vec{\alpha}$-stability with strict inequalities. Moreover, if $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ the $\vec{\alpha}$-polystability condition means

$$\mu_{\vec{\alpha}}(\mathcal{E}) = \mu_{\vec{\alpha}}(\mathcal{E}_1) = \mu_{\vec{\alpha}}(\mathcal{E}_2) = 0.$$

Proposition 3.9. *Let $(\mathcal{E}, \theta)$ be a strictly $\vec{\alpha}$-semi-stable irregular Higgs bundle. Then, there exists a filtration*

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \mathcal{E}_2 = 0$$

by subbundles preserved by θ so that the irregular Higgs bundles induced on the vector bundles

$$\mathcal{E}_0/\mathcal{E}_1, \quad \mathcal{E}_1$$

are $\vec{\alpha}$-stable of the same parabolic slope as $(\mathcal{E}, \theta)$. Moreover, the isomorphism classes of the associated graded irregular Higgs bundles with respect to this filtration are uniquely determined up to reordering.

This filtration is called the ***Jordan–Hölder filtration***, see [21].

Proof. If $\mathcal{F}$ is a destabilizing subbundle then we set $\mathcal{E}_1 = \mathcal{F}$. Then, $\mathcal{Q} = \mathcal{E}/\mathcal{F}$ is a vector bundle because $\mathcal{F}$ is a subbundle rather than just a subsheaf. Stability of the rank-1 irregular Higgs bundles on $\mathcal{F}, \mathcal{Q}$ is obvious. The slope condition immediately follows from additivity of the parabolic degree.

As for uniqueness, assume there exists another filtration

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1' \supset \mathcal{E}_2 = 0$$

satisfying the same properties, with $\mathcal{E}_1' \neq \mathcal{E}_1$. Then, on a Zariski open subset of $\mathbb{C}P^1$ there is a direct sum decomposition

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_1' \tag{6}$$

preserved by θ. The set of parabolic weights of $\mathcal{E}$ is the union of the sets of parabolic weights of $\mathcal{E}_1$ and of $\mathcal{E}_1'$. On the other hand, we clearly have an inclusion of sheaves

$$\mathcal{E} \supseteq \mathcal{E}_1 \oplus \mathcal{E}_1'$$

over $\mathbb{C}P^1$. It follows from the above observation and the equality of parabolic slopes that the algebraic degree of the two sides of this formula agree. We infer that (6) holds over $\mathbb{C}P^1$, in particular the couples of rank-1 Higgs bundles

$$\mathcal{E}_0/\mathcal{E}_1 \cong \mathcal{E}_1', \quad \mathcal{E}_1$$

and

$$\mathcal{E}_0/\mathcal{E}_1' \cong \mathcal{E}_1, \quad \mathcal{E}_1'$$

agree up to transposition. □

Remark 3.10. *The Jordan–Hölder filtration of an $\vec{\alpha}$-stable irregular Higgs bundle $(\mathcal{E}, \theta)$ is defined to be the trivial filtration. Clearly, this filtration also has the property that the associated graded object with respect to it only contains stable objects.*

Definition 3.11. *Let $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ be two semi-stable irregular Higgs bundles of rank 2. We say that they are **S-equivalent** if the associated graded Higgs bundles for their Jordan–Hölder filtrations are isomorphic.*

In particular, if $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ are stable then they are S-equivalent if and only if they are isomorphic.

3.2. **Irregular Dolbeault moduli spaces.** The results of this section hold in (or at least, can be directly generalized to) the case of irregular Higgs bundles of arbitrary rank.

Let us spell out the basic existence results that we will use. These results follow from the work of O. Biquard and Ph. Boalch in the semi-simple case, and from the work of T. Mochizuki in the general case.

Theorem 3.12. *There exists a smooth hyperKähler manifold $\mathcal{M}^s(\vec{\alpha})$ parameterizing isomorphism classes of $\vec{\alpha}$-stable irregular Higgs bundles of the given semi-simple irregular types with fixed parameters.*

Proof. Assume first that the irregular type of $(\mathcal{E}, \theta)$ near the marked points is semi-simple, i.e. that the local forms of θ are given by (20) and (22) in the case (2,2) and by (51) and (53) in the case (3,1). [2, Theorem 5.4] shows that irreducible solutions of Hitchin's equations in certain weighted Sobolev spaces up to gauge equivalence form a smooth hyperKähler manifold. [2, Theorem 6.1] implies that if $\mu_{\vec{\alpha}}(\mathcal{E}) = 0$ then irreducible solutions of Hitchin's equations up to gauge equivalence are in bijection with analytically stable irregular Higgs bundles up to gauge equivalence. Finally, according to [2, Section 7] the category of analytically stable irregular Higgs bundles (with gauge transformations as morphisms) is in equivalence with the groupoid of algebraically stable irregular Higgs bundles, and moreover this equivalence respects the analytic and algebraic $\vec{\alpha}$-stability conditions. This proves the statement in the semi-simple case.

The proof of the general case follows similarly from the existence of a harmonic metric, see [20] and [17, Corollary 16.1.3]. □

It is possible to extend this result slightly in order to take into account all $\vec{\alpha}$-semi-stable irregular Higgs bundles.

Theorem 3.13. *There exists a moduli stack $\mathcal{M}^{ss}(\vec{\alpha})$ parameterizing S-equivalence classes of $\vec{\alpha}$-semi-stable irregular Higgs bundles of the given semi-simple irregular types with fixed parameters.*

Proof. It follows from [2, Theorem 6.1] in the semi-simple case and [20] and [17, Corollary 16.1.3] in general that there exists a compatible Hermitian–Einstein metric for the irregular Higgs bundle $(\mathcal{E}, \theta)$ if and only if it is $\vec{\alpha}$-poly-stable. We deduce that the space

$$\{\vec{\alpha}\text{-polystable } (\mathcal{E}, \theta)\}/\text{gauge equivalence} \tag{7}$$

is the quotient of an infinite-dimensional vector space by the action of an (infinite-dimensional) gauge group. This endows (7) with the structure of a stack in groupoids. The stable locus is treated in Theorem 3.12. It is therefore sufficient to prove:

Lemma 3.14. *There is a bijection between the set of isomorphism classes of strictly $\vec{\alpha}$-polystable irregular Higgs bundles and the set of S-equivalence classes of strictly $\vec{\alpha}$-semi-stable irregular Higgs bundles.*

Proof. According to Proposition 3.9, the map that associates to a strictly $\vec{\alpha}$-semi-stable irregular Higgs bundle the isomorphism class of the associated graded object of its Jordan–Hölder filtration is well-defined. By the definition of S-equivalence, this map factors to an injective map ι from

$$\{\text{strictly } \vec{\alpha}\text{-semi-stable } (\mathcal{E}, \theta)\}/\text{S-equivalence}$$

to

$$\{\text{strictly } \vec{\alpha}\text{-polystable } (\mathcal{E}, \theta)\}/\text{isomorphism}.$$

As any strictly $\vec{\alpha}$-polystable object is also strictly $\vec{\alpha}$-semi-stable, ι is also surjective. □

This concludes the proof of Theorem 3.13. □

3.3. **Irregular Hitchin fibration.** Let us denote by $\mathrm{Tot}(K_C(D))$ the total space of the line bundle $K_C(D)$ and let Z stand for the compactification of $\mathrm{Tot}(K_C(D))$ by one curve at infinity:

$$Z_C(D) = \mathbb{P}_C(K_C(D) \oplus \mathcal{O}_C). \tag{8}$$

The surface $Z_C(D)$ is projective with a natural inclusion of C given by the 0-section of $K_C(D)$. In the case $C = \mathbb{C}P^1$ and (2,2) or (3,1) we have

$$Z_C(D) = \mathbb{F}_2, \tag{9}$$

the Hirzebruch surface of degree 2. We will denote by

$$p\colon Z_C(D) \to C \tag{10}$$

the canonical projection. By an abuse of notation, we will also denote by p the restriction of this projection to any subscheme of $Z_C(D)$. Let ζ denote the canonical section of $p^* K_C(D)$.

Consider an irregular Higgs bundle $(\mathcal{E}, \theta)$ of rank 2. For the identity automorphism $\mathrm{I}_{\mathcal{E}}$ of $\mathcal{E}$ we may consider the characteristic polynomial

$$\chi_\theta(\zeta) = \det(\zeta \mathrm{I}_{\mathcal{E}} - \theta) = \zeta^2 + s_1 \zeta + s_2, \tag{11}$$

where we naturally have

$$s_1 \in H^0(C, K_C(D)), \quad s_2 \in H^0(C, K_C^2(2 \cdot D)).$$

Definition 3.15. *The* ***irregular Hitchin map*** *of $\mathcal{M}^{ss}$ is defined by*

$$\begin{aligned} h\colon \mathcal{M}^{ss}(\vec{\alpha}) &\to H^0(C, K_C(D)) \oplus H^0(C, K_C^2(2 \cdot D)) \\ (\mathcal{E}, \theta) &\mapsto (s_1, s_2). \end{aligned}$$

The curve $Z_{(s_1,s_2)}$ in $Z_C(D)$ with Equation (11) *is called the* ***spectral curve*** *of $(\mathcal{E}, \theta)$.*

For reasons that will become clear in the discussion preceding (29) and (60), we use the simpler notation

$$t = (s_1, s_2). \tag{12}$$

This quantity t is a natural coordinate of the Hitchin base B; the curve $Z_{(s_1,s_2)}$ will be denoted by Z_t.

Theorem 3.16 ([25])**.** *There exists a ruled surface $\widetilde{Z}_C(D)$ birational to $Z_C(D)$ such that the groupoid of irregular Higgs bundles of the given semi-simple irregular types with fixed parameters is isomorphic to the relative Picard groupoid of torsion-free coherent sheaves of rank* 1 *over an open subset in a Hilbert scheme of curves in $\widetilde{Z}_C(D)$.* □

The surface $\widetilde{Z}_C(D)$ can be explicitly described in terms of a sequence of blow-ups, depending on the parameters appearing in the irregular type. The conditions that one needs to impose on the support curves of the torsion-free sheaves on $\widetilde{Z}_C(D)$ are also very explicit. We do not give a detailed description neither of these conditions nor of $\widetilde{Z}_C(D)$ in complete generality, because they involve much notation. We recommend the interested reader to refer to [25]. In the following, we will use Theorem 3.16 in two particular cases, and we will spell out the surface $\widetilde{Z}_C(D)$ and the conditions on torsion-free sheaves resulting from the general construction of [25] only in these cases, see Figures 4-8.

Notation 3.17. *In this paper we will write*

$$X = \widetilde{Z}_C(D).$$

This shorthand is justified because we will consider normalizations of individual fibers X_t of $\widetilde{Z}_C(D)$, traditionally denoted by $\tilde{X}_t$, and we prefer to avoid double tildes.

It follows from Theorem 3.16 that in the semi-simple case the image of h is a Zariski open subset of a linear system B in the complete linear system $L = |rC|$ of curves in $Z_C(D)$. We will show similar statements in some non-semisimple cases, see Lemma 10.1.

Definition 3.18. *For $t = (s_1, s_2) \in B$ the* ***semi-stable Hitchin fiber*** *over t is*

$$\mathcal{M}_t^{ss}(\vec{\alpha}) = h^{-1}(t). \tag{13}$$

The Hitchin fiber over t has a Zariski open subvariety

$$\mathcal{M}_t^{s}(\vec{\alpha}) \subseteq \mathcal{M}_t^{ss}(\vec{\alpha})$$

called the ***stable Hitchin fiber*** *parameterizing stable irregular Higgs bundles in $h^{-1}(t)$.*

If the curve Z_t corresponding to some $t \in B$ is irreducible and reduced (in particular, if it is smooth) then we have

$$\mathcal{M}_t^{s}(\vec{\alpha}) = \mathcal{M}_t^{ss}(\vec{\alpha}).$$

Indeed, as the spectral curve of any sub-object $(\mathcal{F}, \theta_{\mathcal{F}})$ of any $(\mathcal{E}, \theta) \in \mathcal{M}_t^{ss}(\vec{\alpha})$ is a subscheme of Z_t, we see that under the above assumptions any $(\mathcal{E}, \theta) \in \mathcal{M}_t^{ss}(\vec{\alpha})$ is in fact irreducible, hence stable.

3.4. **The Grothendieck ring.** Let $\mathrm{Var}_{\mathbb{C}}$ be the category of algebraic varieties over $\mathbb{C}$. We let $\mathbb{Z}[\mathrm{Var}_{\mathbb{C}}]$ stand for the abelian group of formal linear combinations of varieties with integer coefficients. We introduce a ring structure on $\mathbb{Z}[\mathrm{Var}_{\mathbb{C}}]$ by the defining the product as the Cartesian product. We introduce the equivalence relation $\sim$ on $\mathbb{Z}[\mathrm{Var}_{\mathbb{C}}]$ generated by the following relations: for any variety X and proper closed subvariety $Y \subset X$ we let

$$X \sim (X \setminus Y) + Y.$$

Definition 3.19. *The* ***Grothendieck ring of varieties*** $K_0(\mathrm{Var}_{\mathbb{C}})$ *is the quotient ring*

$$K_0(\mathrm{Var}_{\mathbb{C}}) = \mathbb{Z}[\mathrm{Var}_{\mathbb{C}}]/\sim.$$

The class of an algebraic variety X is denoted by $[X]$.

We use the notation

$$\mathbf{L} = [\mathbb{C}]$$

for the class of the line and

$$\mathbf{1} = [\text{point}]$$

for the class of a point. In particular, we have

$$[\mathbb{C}P^1] = \mathbf{L} + \mathbf{1},$$
$$[\mathbb{C}^\times] = \mathbf{L} - \mathbf{1}.$$

We will be interested in the classes $[\mathcal{M}_t^s(\vec{\alpha})]$ and $[\mathcal{M}_t^{ss}(\vec{\alpha})]$ of the (semi-)stable Hitchin fibers over all points $t \in B$.

4. Elliptic pencils on the Hirzebruch surface $\mathbb{F}_2$

By Equation (9), the ruled surface $Z_C(D)$ can be identified with the second Hirzebruch surface $\mathbb{F}_2$. In this section we will examine pencils on $\mathbb{F}_2$ generated by the following two curves. The curve C_∞ *at infinity* has three components: the section at infinity (the one with homological square -2) with multiplicity two together with two fibers, which have (a) multiplicities two (called the $(2,2)$-case), or (b) one of them is of multiplicity three, the other is of multiplicity one (which is referred to as the $(3,1)$-case). The other curve generating the pencil is disjoint from the section at infinity and intersects the generic fiber twice. Such a curve is called a *double section* of the ruling on the Hirzebruch surface $\mathbb{F}_2$.

A simple homological computation shows that the two curves above are homologous: if S_∞ denotes the homology class of the section at infinity, S_0 is the homology class of the 0-section and F is the homology class of the fiber of the ruling $p: \mathbb{F}_2 \to \mathbb{C}P^1$, then the identity $S_\infty = S_0 - 2F$ implies that the double section and the curve at infinity described above are homologous. Since the homological square of $2S_0$ is eight, the pencil becomes a fibration once we blow up the Hirzebruch surface eight times. Since there are higher order base points in the pencil, we need to apply infinitely close blow-ups. Indeed, in each case there are two, three or four base points. It is a simple fact that the eight-fold blow-up of $\mathbb{F}_2$ (which itself is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$) is diffeomorphic to the rational elliptic surface, that is, the 9-fold blow-up of the projective plane, denoted as $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$.

According to Assumption 1.1, in the following we will consider only those pencils which result in elliptic fibrations; in particular, the pencil should contain a smooth curve. In the above setting this condition is equivalent to requiring that the double section intersects the fiber component(s) of the curve C_∞ at infinity with multiplicity > 1 only in smooth points.

4.1. **Singular fibers in elliptic fibrations.** Singular fibers in an elliptic fibration have been classified by Kodaira [15]. For description of these fibers, see also [8, 22]. In the following we will need only a subset of all potential singular fibers, so we recall only those.

- The *fishtail fiber* (also called I_1) is topologically an immersed sphere with one positive double point.
- The *cusp fiber* (also called II) is a sphere with a single singular point, and the singularity is a cusp singularity (that is, a cone on the trefoil knot).
- The I_n fiber ($n \geq 2$) is a collection of n spheres of self-intersection -2, all with multiplicity one, intersecting each other transversally in a circular manner, as shown by Figure 1. In this paper we will need only the cases when $n = 2, 3$.

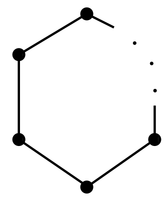

FIGURE 1. Plumbing graph of the singular fiber of type I_n. Dots denote rational curves of self-intersection -2 (and multiplicity one), and the dots are connected if and only if the corresponding curves intersect each other transversally in a unique point. In I_n there are n curves, intersecting along the circular manner shown by the diagram.

- The I_n^*-fiber ($n \geq 0$) contains $n+5$ transversally intersecting (-2)-spheres, as shown by Figure 2(a). We will have fibers of such type for $n = 2, 3, 4$.
- The $\tilde{E}_6$, $\tilde{E}_7$, III and IV fibers all consist of (-2)-spheres intersecting according to the diagrams of Figures 2(c), (d) and 3(a) and (b).

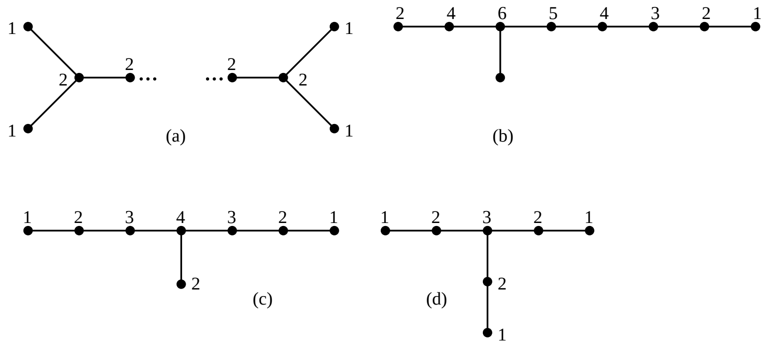

FIGURE 2. Plumbings of singular fibers of types (a) I_n^*, (b) $\tilde{E}_8$, (c) $\tilde{E}_7$, and (d) $\tilde{E}_6$. Integers next to vertices indicate the multiplicities of the corresponding homology classes in the fiber. All dots correspond to rational curves with self-intersection -2. In I_n^* we have a total of $n+5$ vertices; in particular, I_0^* admits a vertex of valency four.

A simple blow-up sequence shows that in case the curve at infinity in the pencil is of type $(2, 2)$ (that is, contains two fibers, each with multiplicity two), then the fibration will have

(1) An I_4^*-fiber if the pencil has two base points;
(2) An I_3^*-fiber if the pencil has three base points;

(3) An I_2^*-fiber if the pencil has four base points.

In more details, the blow-up process can be pictured as in Figures 4 and 5. In the diagram we only picture the blow-up of the base points on one of the fibers of the Hirzebruch surface. The base points of the pencil are smooth points of all the curves other than the curve at infinity, hence we have two cases: when there are two base points on the given fiber (depicted in Figure 4) and when there is a single one (in which case the curves in the pencil are tangent to the fiber of the Hirzebruch surface) — shown by Figure 5. Each case requires 4 (infinitely close) blow-ups. The fibers of the ruling on the Hirzebruch surface $\mathbb{F}_2$ which are part of the curve at infinity (both with multiplicity 2) will be denoted by F_2 and F_2', respectively.

The classification of other singular fibers next to I_2^*, I_3^* or I_4^* reads as follows.

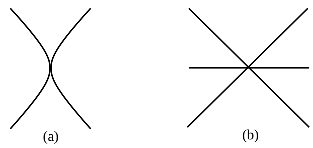

FIGURE 3. Singular fibers of types III and IV in elliptic fibrations. In (a) the two curves are tangent with multiplicity two, and in (b) the three curves pass through one point and intersect each other there transversely.

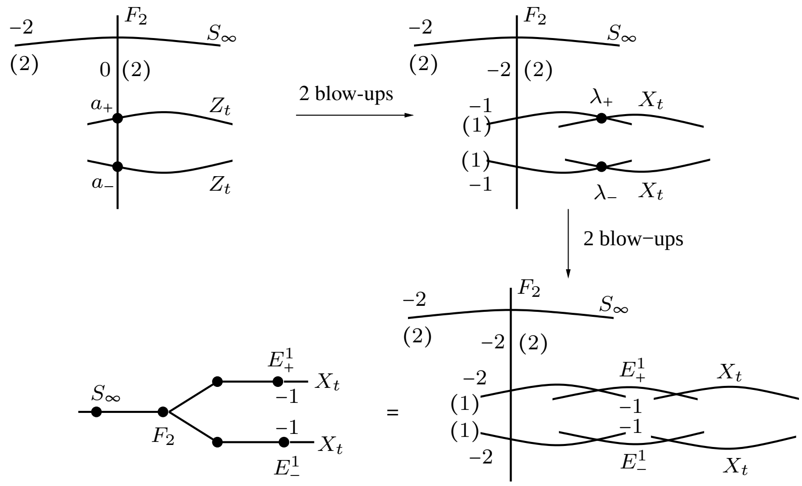

FIGURE 4. The diagram shows the blow-up of the two base points on the fiber F_2 in two steps (4 blow-ups altogether). This case was denoted by (S) in Subsection 2.1 (while (s) is the similar case on the other fiber F_2' of the Hirzebruch surface with multiplicity 2, after substituting $(a_\pm, \lambda_\pm)$ with $(b_\pm, \mu_\pm)$ and $E^1_\pm$ with $E^2_\pm$). The curves are denoted by arcs, the negative numbers next to them are the self-intersections, while the parenthetical positives are the multiplicities in the fiber at infinity. The curve of the pencil (giving rise to the fiber over $t \in B$) is denoted by Z_t and its proper transform is by X_t. (Every proper transform, even in the intermediate steps will be denoted by X_t; hopefully this sloppyness in the notation will not create any confusion.) Solid dots indicate the points where the next blow-up will be applied. We also include the plumbing description of the (relevant part of the) fiber at infinity.

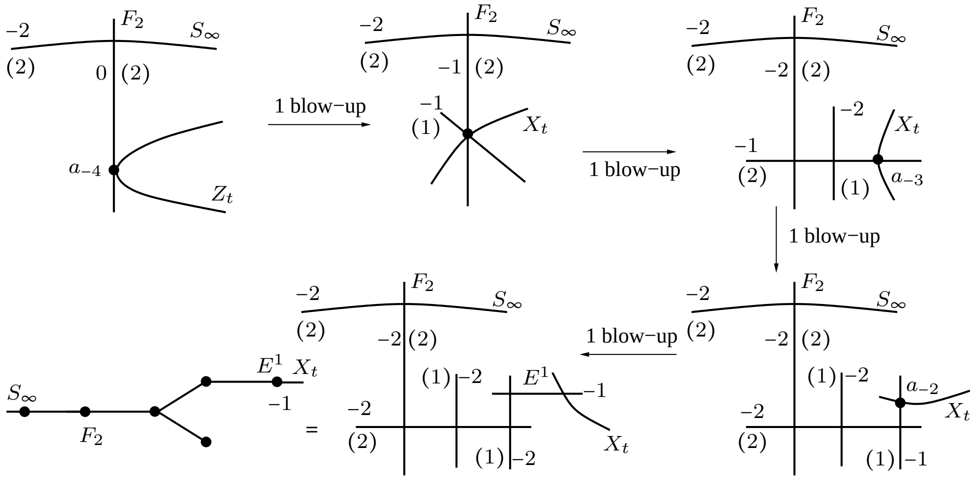

FIGURE 5. In this diagram we blow up the single base point four times. (This is the case denoted by (N) in Subsection 2.1; the corresponding case (n) is given by considering F_2' instead of F_2 and changing a_{-j} to b_{-j} for $j = 4, 3, 2$ and E^1 to E^2.) We use the same conventions as before.

Proposition 4.1 ([16, 19, 22]). *An elliptic fibration on the rational elliptic surface* $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ *with*

- *an* I_4^**-fiber has two further* I_1*-fibers;*
- *an* I_3^**-fiber has further singular fibers which are either (i) three* I_1*-fibers or (ii) an* I_1*-fiber and a fiber of type* II*;*
- *an* I_2^**-fiber has further singular fibers as follows: either (i) four* I_1*-fibers, or (ii) a type* II *and two* I_1*, or (iii) an* I_2 *and two* I_1*, or (iv) two* I_2*, or (v) two type* II*, or (vi) a type* III *and an* I_1*.* □

The similar blow-up sequence as before, now applied to the case $(3, 1)$ (that is, when the curve at infinity has two fiber components, one with multiplicity three, and the other with multiplicity one) will have

(1) an $\tilde{E}_7$-fiber if the fiber with multiplicity three contains a unique base point, and
(2) an $\tilde{E}_6$-fiber if the fiber with multiplicity three contains two base points.

The blow-up sequence in this case is slightly longer, requires the analysis of more cases; these cases will be shown by Figures 6, 7 and 8. Once again, we only depict one of the fibers of the Hirzebruch surface, which is part of the curve at infinity. Since the two multiplicities are different, they need different treatment. The fiber of the Hirzebruch surface in the curve at infinity having multiplicity 3 will be denoted by F_3, while the fiber with multiplicity 1 is F_1.

The classification result in these cases reads as follows:

Proposition 4.2 ([16, 19, 22]). *An elliptic fibration on the rational elliptic surface* $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ *with*

- *an* $\tilde{E}_7$*-fiber has either (i) three* I_1*-fibers, (ii) an* I_2 *and an* I_1*, (iii) a type* II *and an* I_1 *or (iv) a type* III*-fiber.*
- *an* $\tilde{E}_6$*-fiber has either (i) four* I_1*-fibers, (ii) an* I_2 *and two* I_1*, (iii) a type* II *and two* I_1*, (iv) a type* II *and an* I_2*, (v) two type* II*, (vi) an* I_3 *and an* I_1*, (vii) a type* III *and an* I_1*, or (viii) a type* IV *fiber.* □

4.2. **Pencils with sections.** We need to pay special attention to those fibrations which have fibers with more than one component (besides the fiber coming from the curve at infinity

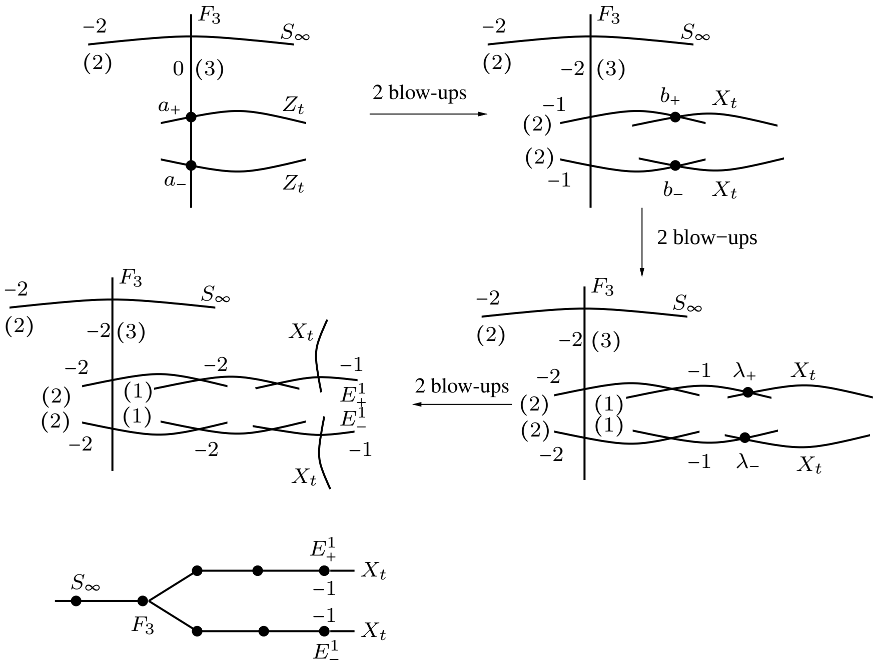

FIGURE 6. The blow-up of the two base points on the fiber F_3 with multiplicity 3 (6 blow-ups altogether). This is the case listed as (S) in Subsection 2.2. We use the same conventions as before.

in the pencil). We say that the pencil *contains a section* if there is a curve in the pencil (other than the curve at infinity) which has more than one component. Notice that if the curve in the pencil has more than one component, then it is the union of two sections of the Hirzebruch surface $\mathbb{F}_2$ (equipped with its $\mathbb{CP}^1$-fibration). Such curves give rise to singular fibers in the elliptic fibration we get after blowing up $\mathbb{F}_2$ eight times, which (according to the classification result recalled above) is either of type III, type IV, I_2 or an I_3 fiber.

In some cases, the existence of certain singular fibers indeed imply the existence of sections in the pencil. Below we list two such cases.

Lemma 4.3. *Consider the second Hirzebruch surface $\mathbb{F}_2$ with a pencil of type $(2,2)$ or $(3,1)$ having four base points. If the elliptic fibration resulting from the blow-up of the pencil has a singular fiber of type III, I_2 or I_3 (besides the type I_2^*-fiber in the $(2,2)$ case or the $\widetilde{E}_6$-fiber in the $(3,1)$ case), then the pencil contains a section.*

Proof. Notice that all these fibers have more than one components. So when blowing them down to $\mathbb{F}_2$, we either get a curve in the pencil with more than one components (i.e., we have a pencil with a section), or one or two components of the fiber must be blown down. In this case, however, we get a curve in the pencil which has a singular point in one of the base points of the pencil. Since we consider only pencils containing at least one smooth curve, this singular point cannot be on the fiber with multiplicity two or three. Since there are two base points on the fiber with multiplicity one in the $(3,1)$ case, both must be smooth points of all the curves in the pencil, hence this second case cannot occur, and we conclude that the pencil contains a section. $\square$

Lemma 4.4. *Consider the second Hirzebruch surface $\mathbb{F}_2$ equipped with a pencil of type $(3,1)$ having three base point, two in the fiber with multiplicity three and one in the fiber with multiplicity one. If the elliptic fibration resulting from the blow-up of the pencil on the*

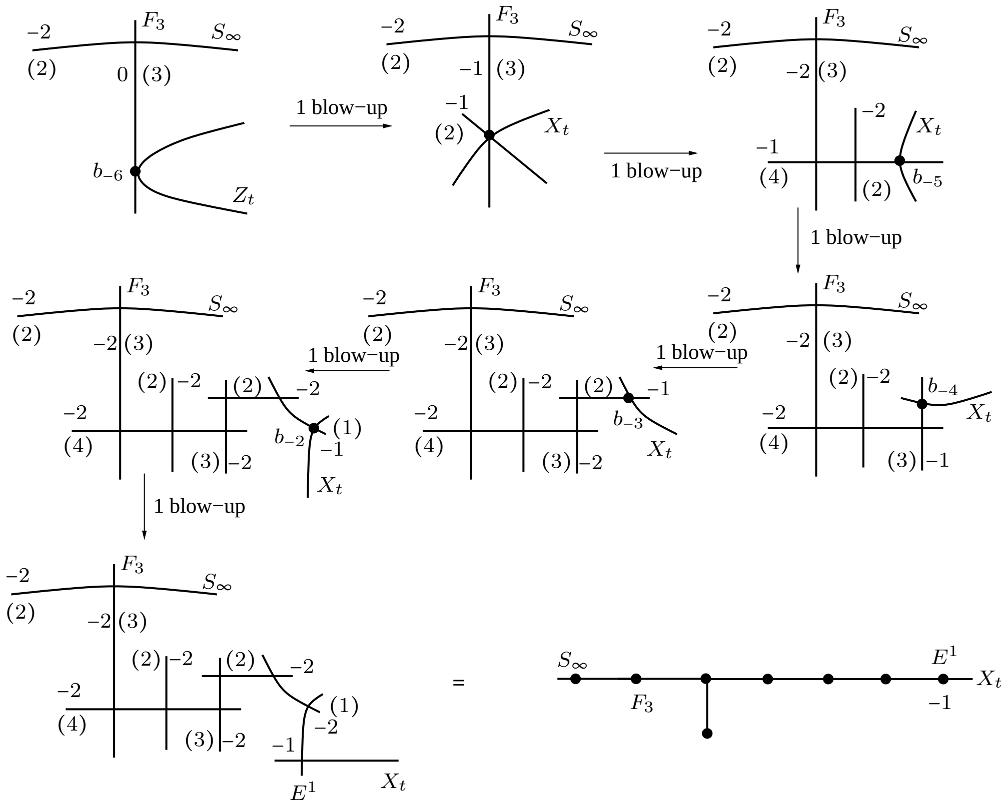

FIGURE 7. In the diagram we blow up the single base point on the fiber F_3 of multiplicity 3 six times. This case corresponds to (N) of Subsection 2.2. We use the same conventions as before.

Hirzebruch surface has a singular fiber (besides the one of type $\widetilde{E}_6$) of type I_3 or IV, then the pencil on the Hirzebruch surface contains a section.

Proof. Consider the blow-down of the I_3 or IV singular fiber. If the result in the pencil is not connected, we found a section in the pencil. If the result is a connected curve, it must have a singularity, which is either a nodal or a cusp singularity. Since by our assumption the pencil contains smooth curves, the singularity must be in the base point on the fiber of multiplicity one. Blowing up this base point once, we get a curve of two components intersecting each other either in two points, or in one point with multiplicity two. Since these intersection points are not base points of the resulting pencil, we will not blow them up again, and so in the resulting fibration we will have such singular fibers. Notice however, that there are no such pairs of curves in an I_3 or a type IV fiber, verifying that this second case is not possible, hence proving the lemma. □

Notice, however, that (as the above proof shows) the fibration can have fibers with more than one components even if the pencil has no section. Indeed, consider the case $(3, 1)$ with a single base point on the fiber of multiplicity one. Suppose first that a curve in the pencil has a singularity at this base point, but it is not a section of the Hirzebruch surface. This implies that the curve is either nodal or cuspidal, and when blowing up the base point once, we get an I_2-fiber in case the double section had a node at the base point, or a type III-fiber in case it had a cusp there. In this case a component of the singular fiber originates from the exceptional curve of one of the blow-ups — a similar phenomenon happens when we have one base point on the fiber with multiplicity one and two on the

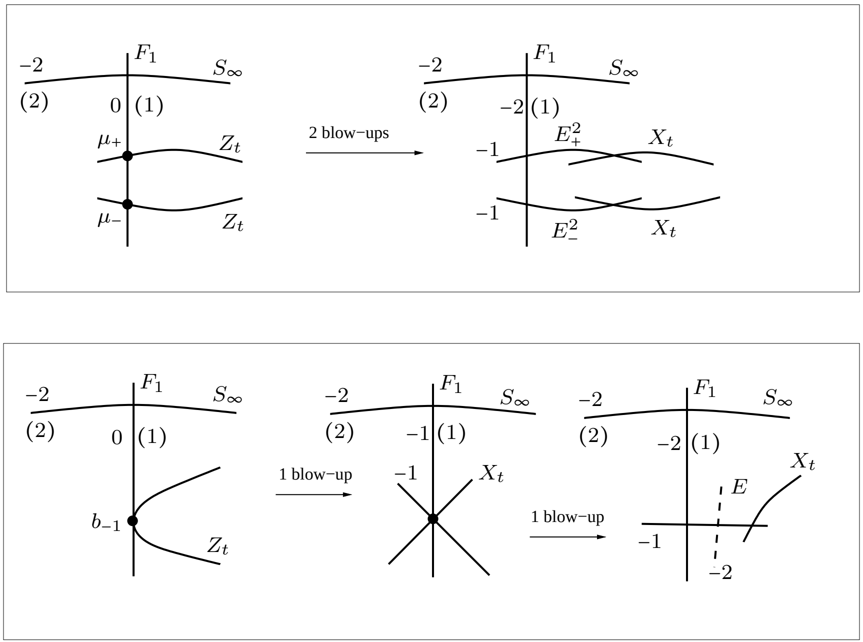

FIGURE 8. The upper diagram shows the blow-up of the two base points on the fiber F_1 of multiplicity 1 (2 blow-ups altogether); this case corresponds to case (s) of Subsection 2.2. In the lower diagram we blow up the single base point on the fiber of multiplicity 1 twice. The dashed curve E is a rational (-2)-curve, which is part of another singular fiber. This is the case which corresponds to (n) of Subsection 2.2.

fiber with multiplicity three and two sections either intersect transversely or are tangent at the base point on the fiber with multiplicity one. In the first case this configuration provides an I_3-fiber, in the second case we get a type IV-fiber.

Indeed, fibers of the elliptic fibration with more than one components are always present in some cases:

Lemma 4.5. *Suppose that we are in case the* $(3,1)$ *and on the fiber with multiplicity one we have a single base point. Then there is a fiber (besides the one originated from the curve at infinity, which is either* $\widetilde{E}_6$ *or* $\widetilde{E}_7$*) which contains a* (-2)*-sphere that maps to a point under* $\mathfrak{p}$*, where* $\mathfrak{p}$ *is the ruling, introduced in Equation* (10).

Proof. Consider a double section C in the pencil. If the base point P on the fiber of multiplicity one is a singular point of this curve, then the above mentioned blow-up shows that the fiber coming from C has a component which is a (-2)-sphere.

Assume now that P is a smooth point of C. Since in this case C is tangent to the fiber, we need to apply two infinitely close blow-ups, and the exceptional divisor of the first blow-up will become a (-2)-sphere in one of the fibers of the resulting fibration, see the dashed curve E in the lower diagram of Figure 8. □

We will need one further result similar to the previous ones:

Lemma 4.6. *Suppose that we are in case the* $(3,1)$*, and on the fiber with multiplicity one we have two base points, while on the fiber with multiplicity three we have one. Then the fiber originating from the curve at infinity is of type* $\widetilde{E}_7$*, and we cannot have a type* III *or* I_2 *fiber next to it.*

Proof. It is a simple calculation to check that the fiber at infinity is of type $\widetilde{E}_7$, cf. Figure 7 and the upper part of Figure 8. It is easy to see that the pencil cannot contain a section, since the corresponding curve would admit a singularity at the base point on the fiber with multiplicity three, hence the pencil would not contain smooth curves. Therefore, a fiber of type III or I_2 would originate from a connected curve, hence when blowing curves back down, one component of the fiber must be blown down. This means that the other component has a singularity at one of the base points. This is not possible in the base point on the fiber of multiplicity three (since we assume the existence of a smooth curve in the pencil). Similarly, the two base points on the fiber of multiplicity one must be also smooth points of all curves, otherwise the intersection multiplicity with that fiber would rise to at least three. This concludes the argument. □

A certain converse of the above lemmas also holds:

Lemma 4.7. *Suppose that we are in case the* $(2,2)$ *or* $(3,1)$*. If the pencil contains a section, then the elliptic fibration on the 8-fold blow up will contain a fiber of type* I_2, I_3, III *or* IV*.*

Proof. Assume first that the pencil has four base points. Then the base points are necessarily smooth points of each curve in the pencil. The two components of the reducible curve in the pencil can meet transversally (in which case the fibration will have an I_2 fiber) or can be tangent to each other (when the fibration will have a type III fiber).

Suppose that there is a fiber of the Hirzebruch surface containing a single base point, and the pencil contains a section. By our assumption (that the generic curve in the pencil is smooth) this assumption implies that the fiber in question is of multiplicity one, hence we need to examine two further possibilities: we must be in case (3,1) and the two components of the reducible curve in the pencil either meet transversely in the unique base point on the fiber with multiplicity one, or the two curves are tangent there. In the first case we will have a fiber of type I_3 in the pencil, and in the second case we will get a fiber of type IV, verifying the claim of the lemma. □

5. THE ORDER OF POLES ARE 2 AND 2

After these preliminaries, now we start proving the results announced in Section 1. In this section we will discuss the complex surfaces relevant for the cases of Theorems 2.1, 2.2 and 2.3. Recall from Notation 3.17 the definition of the complex surface X. We will prove:

Proposition 5.1. *Assume that the polar part of the Higgs field is regular semisimple near* q_1 *and regular semisimple near* q_2*. Then* X *is biregular to the complement of the fiber at infinity (of type* I_2^**) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) *if* $\Delta = 0$ *and* $L^2 = M^2 \neq 0$*, then a type* III *and an* I_1 *fibers;*
(2) *if* $\Delta = 0$*,* $L^2 = -M^2 \neq 0$ *and* $M^3 = 8ABL$*, then two type* II *fibers;*
(3) *if* $\Delta = 0$*,* $L^2 = -M^2 \neq 0$ *and* $M^3 \neq 8ABL$*, then a type* II *and two* I_1 *fibers;*
(4) *if* $\Delta = 0$ *and* $L^2 \neq \pm M^2$*, then a type* II *and two* I_1 *fibers again;*
(5) *if* $\Delta \neq 0$ *and* $L = M = 0$*, then two* I_2 *fibers;*
(6) *if* $\Delta \neq 0$ *and* $L^2 = M^2 \neq 0$*, then a* I_2 *and two* I_1 *fibers ;*
(7) *if* $\Delta \neq 0$ *and* $L^2 \neq M^2$*, then four* I_1 *fibers ;*

for Δ *see (35). The case* $\Delta = 0$ *and* $L = M = 0$ *implies either* $A = 0$ *or* $B = 0$*, and it does not give an elliptic fibration.*

The above statement is summarized by Table 1.

Proposition 5.2. *Assume that the polar part of the Higgs field is regular semisimple near* q_1 *and non-semisimple near* q_2 *(or vice versa). Then* X *is biregular to the complement of*

	$\mathbf{\Delta = 0}$	$\mathbf{\Delta \neq 0}$
$\mathbf{L = M = 0}$	not semisimple	$2I_2$
$\mathbf{L^2 = M^2 \neq 0}$	$III + I_1$	$I_2 + 2I_1$
$\mathbf{L^2 = -M^2 \neq 0}$ and $\mathbf{M^3 = 8ABL}$	$2II$	$4I_1$
$\mathbf{L^2 = -M^2 \neq 0}$ and $\mathbf{M^3 \neq 8ABL}$	$II + 2I_1$	
$\mathbf{L^2 \neq \pm M^2}$		

TABLE 1. The type of singular curves in (2,2) case with four base points. We list the singular fibers next to the I_2^* fiber.

the fiber at infinity (of type I_3^*) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:

(1) if $\Delta = 0$, then a type II and an I_1 fibers;
(2) if $\Delta \neq 0$, then three I_1 fibers;

where $\Delta = 4A^3 b_{-3}\left(2L^3 - 27Ab_{-3}\right)$ in case (Sn) (or $\Delta = 4B^3 a_{-3}\left(2M^3 - 27Ba_{-3}\right)$ in case of (Ns)). If $b_{-3} = 0$ (or $a_{-3} = 0$) then the fibration is not elliptic.

Proposition 5.3. *Assume that the polar part of the Higgs field is non-semisimple near q_1 and non-semisimple near q_2. Then X is biregular to the complement of the fiber at infinity (of type I_4^*) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $a_{-3}b_{-3} \neq 0$, then two type I_1 fibers.

If $a_{-3} = 0$ or $b_{-3} = 0$ then the fibration is not elliptic.

Remark 5.4. *Indeed, all implications in the above propositions are if-and-only-if statements. We will prove implications in both directions.*

5.1. **Local form of irregular Higgs bundles.** Our investigation is based on the local description of spectral curves on the Hirzebruch surface of degree 2. Introduce two local charts on $C = \mathbb{C}P^1$: U_1 with $z_1 \in \mathbb{C}$ (and $\{z_1 = 0\} = q_1$) and U_2 with $z_2 \in \mathbb{C}$ (and $\{z_2 = 0\} = q_2$). In the case (2,2) the line bundle is $K_{\mathbb{C}P^1}(2\cdot\{q_1\} + 2\cdot\{q_2\})$. The bundle $K_{\mathbb{C}P^1}(2\cdot\{q_1\} + 2\cdot\{q_2\})$ admits the trivializing sections κ_i over U_i:

$$\kappa_1 = \frac{\mathrm{d}z_1}{z_1^2},$$
$$\kappa_2 = \frac{\mathrm{d}z_2}{z_2^2}.$$

The conversion from κ_1 to κ_2 is the following:

$$\kappa_1 = \frac{\mathrm{d}z_1}{z_1^2} = -\mathrm{d}z_2 = -z_2^2\kappa_2. \tag{14}$$

The trivialization κ_i induces a trivialization κ_i^2 on $K_C(2\cdot\{q_1\} + 2\cdot\{q_2\})^{\otimes 2}$, $i = 1, 2$.

The Hirzebruch surface can be covered by four charts. We will need only two of those, since we only conisder curves disjoint from the section at infinity (which is a component of the curve C_∞ at infinity). Let us denote $V_i \subset p^{-1}(U_i)$ the complement of the section at infinity in $p^{-1}(U_i)$ $(i = 1, 2)$. Let $\zeta \in \Gamma\left(\mathbb{F}_2, p^*K_C(D)\right)$ be the canonical section, and introduce $w_i \in \Gamma(V_i, \mathcal{O})$ by

$$\zeta = w_i \otimes \kappa_i.$$

Use (14) for the conversion between w_1 to w_2:

$$w_2 \otimes \kappa_2 = \zeta = w_1 \otimes \kappa_1 = -z_2^2 w_1 \otimes \kappa_2.$$

Consider an irregular Higgs bundle $(\mathcal{E}, \theta)$ in the κ_i trivializations ($i = 1, 2$). The local forms of θ near q_1 or q_2 are the following

$$\theta = \sum_{n \geq -2} A_n z_1^n \otimes \mathrm{d}z_1 \quad \text{or} \quad \theta = \sum_{n \geq -2} B_n z_2^n \otimes \mathrm{d}z_2, \tag{15}$$

where $A_n, B_n \in \mathfrak{gl}(2, \mathbb{C})$. Take the characteristic polynomial in Equation (11)

$$\chi_\theta(\zeta) = \det(\zeta \mathrm{I}_{\mathcal{E}} - \theta) = \zeta^2 + s_1 \zeta + s_2, \tag{16}$$

for some

$$s_1 \in H^0(\mathbb{C}P^1, K(2 \cdot \{q_1\} + 2 \cdot \{q_2\})), \quad s_2 \in H^0(\mathbb{C}P^1, K(2 \cdot \{q_1\} + 2 \cdot \{q_2\})^{\otimes 2}).$$

This means that s_1 is a meromorphic differential and s_2 is a meromorphic quadratic differential.

Let us set $\vartheta_1 = \sum_{n \geq 0} A_{n-2} z_1^n$ and $\vartheta_2 = \sum_{n \geq 0} B_{n-2} z_2^n$, so that we have

$$\theta = \vartheta_i \otimes \kappa_i.$$

If we divide by κ_i in (16), then the characteristic polynomial may be rewritten as

$$\chi_{\vartheta_i}(w_i) = \det(w_i \mathrm{I}_{\mathcal{E}} - \vartheta_i) = w_i^2 + w_i f_i + g_i, \tag{17}$$

with

$$s_1 = f_i \kappa_i, \quad s_2 = g_i \kappa_i^2 \quad (i = 1, 2).$$

Now, as $K(2 \cdot \{q_1\} + 2 \cdot \{q_2\}) \cong \mathcal{O}(2)$, the coefficients f_1 and g_1 in the κ_1 trivialization are polynomials in z_1 of degree 2 and 4, respectively:

$$\begin{aligned} f_1(z_1) &= -(p_2 z_1^2 + p_1 z_1 + p_0), \\ g_1(z_1) &= -(q_4 z_1^4 + q_3 z_1^3 + q_2 z_1^2 + q_1 z_1 + q_0), \end{aligned}$$

where all coefficients are elements of $\mathbb{C}$.

Similarly, in the κ_2 trivialization using the formula (14) we get

$$\begin{aligned} f_2(z_2) &= p_0 z_2^2 + p_1 z_2 + p_2, \\ g_2(z_2) &= -\left(q_0 z_2^4 + q_1 z_2^3 + q_2 z_2^2 + q_3 z_2 + q_4\right). \end{aligned}$$

Finally, (17) gives two polynomials in variables z_1, w_1 or z_2, w_2. These polynomials are the local forms of spectral curves in $Z_C(D)$:

$$\chi_{\vartheta_1}(z_1, w_1) = w_1^2 - \left(p_2 z_1^2 + p_1 z_1 + p_0\right) w_1 - \left(q_4 z_1^4 + q_3 z_1^3 + q_2 z_1^2 + q_1 z_1 + q_0\right), \tag{18}$$

$$\chi_{\vartheta_2}(z_2, w_2) = w_2^2 + \left(p_0 z_2^2 + p_1 z_2 + p_2\right) w_2 - (q_0 z_2^4 + q_1 z_2^3 + q_2 z_2^2 + q_3 z_2 + q_4). \tag{19}$$

On the other hand, the spectral curve has an expansion near q_1 and q_2 in which these parameters have a geometric meaning: the matrices A_n and B_n $(n = -2, -1)$ in (15) encode the base locus and the slope of the tangent line of a pencil. As indicated in Subsection 2.1 the letters $(S), (N), (s), (n)$ refer to the following cases which may occur independently of each other.

Near $z_1 = 0$:

(S) the semisimple case

$$\theta = \left[\begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} z_1^{-2} + \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} z_1^{-1} + O(1) \right] \otimes \mathrm{d}z_1, \tag{20}$$

(N) the non-semisimple case

$$\theta = \left(\begin{pmatrix} a_{-4} & 1 \\ 0 & a_{-4} \end{pmatrix} z_1^{-2} + \begin{pmatrix} 0 & 0 \\ a_{-3} & a_{-2} \end{pmatrix} z_1^{-1} + O(1) \right) \otimes \mathrm{d}z_1. \tag{21}$$

Near $z_2 = 0$:

(s) the semisimple case

$$\theta = \left[\begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix} z_2^{-2} + \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix} z_2^{-1} + O(1) \right] \otimes \mathrm{d}z_2, \tag{22}$$

(n) the non-semisimple case

$$\theta = \left(\begin{pmatrix} b_{-4} & 1 \\ 0 & b_{-4} \end{pmatrix} z_2^{-2} + \begin{pmatrix} 0 & 0 \\ b_{-3} & b_{-2} \end{pmatrix} z_2^{-1} + O(1) \right) \otimes \mathrm{d}z_2. \tag{23}$$

Since $\operatorname{tr}\theta$ is a meromorphic function, the residue theorem implies

$$\operatorname{tr}\operatorname{Res}_{z_1=0}\theta + \operatorname{tr}\operatorname{Res}_{z_1=\infty}\theta = 0.$$

According to the various cases regarding the semisimplicity or non-semisimplicity of the local forms at the two poles, this implies

$$\lambda_+ + \lambda_- + \mu_+ + \mu_- = 0, \tag{24a}$$

$$\lambda_+ + \lambda_- + b_{-2} = 0, \tag{24b}$$

$$\mu_+ + \mu_- + a_{-2} = 0, \tag{24c}$$

$$a_{-2} + b_{-2} = 0. \tag{24d}$$

Here and in what follows, for reasons of symmetry we do not consider the case where the local form at $z_1 = 0$ is non-semisimple and at $z_2 = 0$ is semisimple.

The roots of the characteristic polynomial (17) in w_i have Puiseux expansions with respect to z_i. The first several terms of these expansions are equal to the eigenvalues of the matrices (20) – (23). In concrete terms:

(S) The polar part of θ near q_1 has semisimple leading-order term. The series of the 'negative' root of $\chi_{\vartheta_1}(z_1, w_1)$ in (18) up to first order is equal to $a_- + \lambda_- z_1$ and the 'positive' root up to first order is equal to $a_+ + \lambda_+ z_1$. Hence we get the equations

$$a_- = \frac{p_0}{2} - \frac{1}{2}\sqrt{p_0^2 + 4q_0}, \tag{25a}$$

$$\lambda_- = \frac{p_1\sqrt{p_0^2 + 4q_0} - p_0 p_1 - 2q_1}{2\sqrt{p_0^2 + 4q_0}}, \tag{25b}$$

$$a_+ = \frac{p_0}{2} + \frac{1}{2}\sqrt{p_0^2 + 4q_0}, \tag{25c}$$

$$\lambda_+ = \frac{p_1\sqrt{p_0^2 + 4q_0} + p_0 p_1 + 2q_1}{2\sqrt{p_0^2 + 4q_0}}. \tag{25d}$$

(N) The polar part of θ near q_1 has non-semisimple leading-order term. This means that the polynomial $\chi_{\vartheta_1}(z_1, w_1)$ has one ramified root w_1 with branch point $z_1 = 0$. This leads to the formula $p_0^2 + 4q_0 = 0$. We simplify the roots of $\chi_{\vartheta_1}(z_1, w_1)$ using this condition. The Puiseux series of two roots up to first order are equal to the series of eigenvalues of the matrix (21) up to first order. Notice that the expansion allows half integer powers of the variable z_1. The resulting equations are:

$$a_{-4} = \frac{p_0}{2}, \tag{26a}$$

$$\sqrt{a_{-3}} = \frac{1}{2}\sqrt{2p_0 p_1 + 4q_1}, \tag{26b}$$

$$\frac{a_{-2}}{2} = \frac{p_1}{2}. \tag{26c}$$

(s) The polar part of θ near q_2 has semisimple leading-order term. The series of the 'negative' root of $\chi_{\vartheta_2}(z_2, w_2)$ in (19) up to first order is equal to $b_- + \mu_- z_2$ and similarly the 'positive' root up to first order is equal to $b_+ + \mu_+ z_2$. The equations are:

$$b_- = -\frac{p_2}{2} - \frac{1}{2}\sqrt{p_2^2 + 4q_4}, \tag{27a}$$

$$\mu_- = -\frac{p_1\sqrt{p_2^2 + 4q_4} + p_1 p_2 + 2q_3}{2\sqrt{p_2^2 + 4q_4}}, \tag{27b}$$

$$b_+ = -\frac{p_2}{2} + \frac{1}{2}\sqrt{p_2^2 + 4q_4}, \tag{27c}$$

$$\mu_+ = -\frac{p_1\sqrt{p_2^2 + 4q_4} - p_1 p_2 - 2q_3}{2\sqrt{p_2^2 + 4q_4}}. \tag{27d}$$

(n) The polar part of θ near q_2 has non-semisimple leading-order term, that requires the polynomial $\chi_{\vartheta_2}(z_2, w_2)$ to have one ramified root w_2 with branch point $z_2 = 0$. This leads to the formula $p_2^2 + 4q_4 = 0$. We use this condition to simplify the roots of $\chi_{\vartheta_2}(z_2, w_2)$. The series of the roots of $\chi_{\vartheta_2}(z_2, w_2)$ up to first order are equal to the series of eigenvalues of the matrix (23) up to first order. The expansions also allow half integer powers of z_2 The corresponding terms are:

$$b_{-4} = -\frac{p_2}{2}, \tag{28a}$$

$$\sqrt{b_{-3}} = \frac{1}{2}\sqrt{2p_1 p_2 + 4q_3}, \tag{28b}$$

$$\frac{b_{-2}}{2} = -\frac{p_1}{2}. \tag{28c}$$

Now, fix the polar part of θ near the points q_1 and q_2. The polar part near q_1 is independent of the polar part near q_2. This means, that the choice between (S) and (N), and the choice between (s) and (n) can be done independently. Thus we have four possibilities regarding the local behavior of Higgs field: $(Ss), (Sn), (Ns), (Nn)$.

Since the above equations do not depend on q_2 (the coefficient of $g_1(z_1)$ and $g_2(z_2)$), we set

$$t = q_2. \tag{29}$$

Equations (20)–(23) have determined the coefficients of s_1 and s_2 in any possible choice. The given complex parameters (i. e. $a_\pm$, $\lambda_\pm$ etc.) define the pencil of spectral curves of $(\mathcal{E}, \theta)$ parametrized by t.

According to the introduction in Section 4, the pencil χ_θ gives rise to an elliptic fibration in $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ with some singular fibers. Let us denote the spectral curves by $\chi_{\vartheta_1}(z_1, w_1, t)$ in the κ_1 trivialization, and by $\chi_{\vartheta_2}(z_2, w_2, t)$ in the κ_2 trivialization.

According to the remark before Subsection 4.1, no curve in the pencil has a singular point on the fiber component of the curve C_∞ at infinity with multiplicity 2, thus it is sufficient to consider the κ_1 trivialization, i. e. the chart (z_1, w_1). For identifying the singular fibers in the pencil, we look for triples (z_1, w_1, t) such that (z_1, w_1) fits the curve with parameter t and the partial derivatives below vanish:

$$\chi_{\vartheta_1}(z_1, w_1, t) = 0, \tag{30a}$$

$$\frac{\partial \chi_{\vartheta_1}(z_1, w_1, t)}{\partial w_1} = 0, \tag{30b}$$

$$\frac{\partial \chi_{\vartheta_1}(z_1, w_1, t)}{\partial z_1} = 0. \tag{30c}$$

These triples are in one-to-one correspondence with singular points in singular fibers. Every spectral curve Z_t is a double section of the ruling on the Hirzebruch surface $\mathbb{F}_2$, thus every triple (z_1, w_1, t) satisfying Equations (30) maps to distinct points under the ruling p.

Indeed, if one fiber (with fixed t value) contains two singular points with the same z_1 coordinate then the corresponding fiber of p would intersect Z_t with multiplicity higher than two. Furthermore, it does not happen that two singular points with the same z_1 coordinate lies on distinct fibers (two distinct t values): we will see in Equations (33), (41) and (43) that the t values are determined by the z_1 values. Consequently the z_1-values from triples (z_1, w_1, t) are in one-to-one correspondence with singular points.

Before the discussion of the individual cases, we prove a useful lemma about sections.

Lemma 5.5. *Consider the second Hirzebruch surface $\mathbb{F}_2$ with a pencil of type $(2,2)$ having four base points, two in each fiber with multiplicity two. The pencil contains a section if and only if $L \pm M = 0$.*

Proof. If the pencil contains a section, then there is a section

$$\theta \in \Gamma(\mathcal{E}nd(\mathcal{E}) \otimes K_C(2 \cdot \{q_1\} + 2 \cdot \{q_2\}))$$

whose spectral curve has two components. Denote the components by X_+ and X_-. Each of these components passes through exactly two base points. $X_\pm$ are locally the graphs of sections, which in the κ_i trivialization (by Equations (20) and (22)) are

$$\theta_{\pm,1} = (a_\pm + \lambda_\pm z_1 + \dots)\kappa_1,$$
$$\theta_{\pm,2} = (b_\pm + \mu_\pm z_1 + \dots)\kappa_2.$$

The conversion $z_1 = z_2^{-1}$ and Equation (14) imply

$$\mu_+ = -\lambda_\pm,$$
$$\mu_- = -\lambda_\mp.$$

These equations imply $L \pm M = 0$.

Conversely, if $L \pm M = 0$ then the above equations hold, and the sections $\theta_\pm$ satisfy the conditions. □

5.2. **The discussion of cases appearing in Proposition 5.1.** We assume that the polar part of the Higgs field is semisimple near q_1 and semisimple near q_2. Namely, the spectral curve Z_t and the pencil (specified by this spectral curve, together with the curve at infinity) are determined by Equations (25) and (27). The base locus of the pencil consists of four points: $(0, a_-)$ and $(0, a_+)$ in the chart (z_1, w_1) and $(0, b_-)$ and $(0, b_+)$ in the chart (z_2, w_2). According to the list before Proposition 4.1, the fibration has a singular fiber of type I_2^*.

Express the coefficients p_i and q_j from equations listed in cases (S) and (s) above, $i = 0, 1, 2$ and $j = 0, 1, 3, 4$. The characteristic polynomial (17) in the κ_1 and κ_2 trivializations becomes

$$\begin{aligned}\chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 + \big((b_- + b_+) z_1^2 - (\lambda_- + \lambda_+) z_1 - (a_- + a_+)\big) w_1 + b_- b_+ z_1^4 + \\ &+ (b_+\mu_- + b_-\mu_+) z_1^3 - t z_1^2 + (a_+\lambda_- + a_-\lambda_+) z_1 + a_- a_+,\end{aligned} \tag{31}$$

$$\begin{aligned}\chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + \big((a_- + a_+) z_2^2 + (\lambda_- + \lambda_+) z_2 - (b_- + b_+)\big) w_2 + a_- a_+ z_2^4 + \\ &+ (a_+\lambda_- + a_-\lambda_+) z_2^3 - t z_2^2 + (b_+\mu_- + b_-\mu_+) z_2 + b_- b_+.\end{aligned} \tag{32}$$

It is enough to analyze the pencil $\chi_{\vartheta_1}(z_1, w_1, t)$. Consider Equations (30) to determine the singular points. We express w_1 and t from the second and the third equations by z_1.

$$\begin{aligned} w_1(z_1) =& \frac{1}{2}\big((-b_- - b_+) z_1^2 + (\lambda_- + \lambda_+) z_1 + a_- + a_+\big), \\ t(z_1) =& \frac{1}{4z_1}\big(-2(b_- - b_+)^2 z_1^3 + 3(b_+(\lambda_- + \lambda_+ + 2\mu_-) + b_-(\lambda_- + \lambda_+ + 2\mu_+)) z_1^2 + \\ &+ \big(2(a_- + a_+)(b_- + b_+) - (\lambda_- + \lambda_+)^2\big) z_1 - (a_- - a_+)(\lambda_- - \lambda_+)\big).\end{aligned} \tag{33}$$

Now, we substitute the resulting expressions into the first equation and get

$$\begin{aligned}0 &= (b_- - b_+)^2 z_1^4 - (b_+(\lambda_- + \lambda_+ + 2\mu_-) + b_-(\lambda_- + \lambda_+ + 2\mu_+)) z_1^3 + \\ &+ (a_+ - a_-)(\lambda_- - \lambda_+) z_1 - (a_- - a_+)^2.\end{aligned}$$

We can rewrite the equation with the notation of (3) and use Condition (24a):

$$0 = B^2 z_1^4 + BM z_1^3 - ALz_1 - A^2. \tag{34}$$

The roots of this polynomial correspond to the z_1 values of singular points in the singular curves on the Hirzebruch surface $\mathbb{F}_2$, which become fibers on the 8-fold blow up. Since this is a degree-4 polynomial, generally we get four distinct roots, and this corresponds to the fact that there are at most four singular fibers in the fibration.

The quartic polynomial of (34) with variable z_1 has multiple roots if and only if its discriminant

$$A^3 B^3 \left(192A^2B^2LM - 256A^3B^3 - 3AB\left(9L^4 - 2L^2M^2 + 9M^4\right) + 4L^3M^3\right)$$

vanishes.

Lemma 5.6. *The cases $A = 0$ (i. e. $a_- = a_+$) and $B = 0$ (i. e. $b_- = b_+$) lead to the non-regular semisimple case and does not give an elliptic fibration.*

Proof. Let us consider the case $A = 0$, consider the curves of the pencil $\chi_{\vartheta_1}(z_1, w_1, t)$ and substitute a_- with a_+ in the Equation (31). Compute the tangents of $\chi_{\vartheta_1}(z_1, w_1, t)$ at $(z_1 = 0, w_1 = a_+)$ as the implicit derivative of $\chi_{\vartheta_1}(z_1, w_1, t)$ in the point $(0, a_+)$:

$$\frac{\frac{\partial \chi_{\vartheta_1}}{\partial w_1}}{\frac{\partial \chi_{\vartheta_1}}{\partial z_1}} = -\frac{2}{\lambda_- + \lambda_+}.$$

Since $\lambda_- + \lambda_+ \neq \infty$, both branches of the curves intersect the $z_1 = 0$ axis transversely in the point $(0, a_+)$. Therefore this is a singular point of all curves in the pencil, consequently the pencil has no smooth curves, hence the resulting fibration is not elliptic. (See remark before Subsection 4.1.) □

According to Assumption 1.1, we have $A \neq 0$, $B \neq 0$ and we define

$$\Delta = 192A^2B^2LM - 256A^3B^3 - 3AB\left(9L^4 - 2L^2M^2 + 9M^4\right) + 4L^3M^3. \tag{35}$$

The further expressions below are connected to the fact whether the quartic has double, triple or quadruple roots.

$$\begin{aligned}\Delta_0 &= 3AB(LM - 4AB), \\ \Delta_1 &= B^4\left(16ABLM - 64A^2B^2 - 3M^4\right).\end{aligned}$$

5.2.1. *One root.* The quartic polynomial of (34) has one root if and only if $\Delta = \Delta_0 = \Delta_1 = 0$. Simplify Δ_1 using $\Delta_0 = 0$ to get $M = 0$. Setting $M = 0$ in Δ_0 provides that A or B is equal to zero. This is a contradiction, hence this case does not occur.

5.2.2. *Two roots (one triple root).* First we consider if Δ and Δ_0 vanish. Apply $\Delta_0 = 0$ to simplify $\Delta = 0$:

$$0 = -\frac{27}{4} LM \left(L^2 - M^2\right)^2.$$

We have $L \neq 0$ because otherwise $\Delta_0 = 0$ becomes $-12A^2B^2 = 0$ and this is not the regular semisimple case. For the same reason $M \neq 0$, thus $L^2 = M^2 \neq 0$. From Δ_0 we get $B = \frac{LM}{4A}$. Substituting B and L to the quartic (34) and solving it we get a triple root $\mp\frac{2A}{M}$ and a simple root $\pm\frac{2A}{M}$. The plus or minus sign depends on the sign in $L = \pm M$. Hence the fibration has two singular fibers. Proposition 4.1 provides that the fibers are either $III + I_1$ or $2II$. By Lemma 5.5, $L = \pm M$ means that the pencil has a section and Lemma 4.7 shows the existence of fiber of type III.

In the other direction we suppose the fibration has a type III fiber and a fishtail. There are two singular points, i.e. the quartic has two roots. Consequently the discriminant vanishes and by Lemma 4.3 there is a section. This means that $L = \pm M$, or equivalently $L^2 = M^2$.

Moreover $\Delta = 0$ and $L = \pm M$ guarantee $LM = 4AB$ and this leads to the appearance of a triple root and $L \neq 0$.

5.2.3. *Two roots (two double roots).* In the second case, the quartic has two double roots, that is, the shape of the quartic equation is

$$0 = c_1(z - c_2)^2(z - c_3)^2,$$

where $c_i \in \mathbb{C}$ and $i = 1, 2, 3$. Expand the equation and denote the coefficients by r_0, r_1, r_2, r_3, r_4 in ascending order. There are two relations among the coefficients:

$$\begin{aligned} 64r_4^3 r_0 &= \left(r_3^2 - 4r_2r_4\right)^2, \\ r_3^3 + 8r_1r_4^2 &= 4r_2r_3r_4. \end{aligned}$$

Replace r_i by the coefficients of the quartic (34) and get

$$\begin{aligned} 0 &= 64A^2B^2 + M^4, \\ 0 &= 8AB^2L - BM^3. \end{aligned}$$

Solve these equations for B and L with the assumptions $A \neq 0$ and $B \neq 0$.

$$\begin{aligned} B &= \pm\frac{iM^2}{8A}, \\ L &= \mp iM. \end{aligned}$$

Since $L \pm M \neq 0$, the pencil has no section. Equivalently, the quartic has two double roots if and only if

$$M^3 = 8ABL \quad \text{and} \quad L^2 = -M^2. \tag{36}$$

It is easy to see that $\Delta = \Delta_1 = 0$ and $\Delta_0 \neq 0$. In particular, if $L = M = 0$ then $\Delta = -256A^3B^3$, a contradiction since $A \neq 0$ and $B \neq 0$. Thus $L^2 = -M^2 \neq 0$.

Substituting $L = \pm iM$ into Δ, it becomes

$$-4\left(8AB \pm iM^2\right)^2\left(AB \mp iM^2\right).$$

The condition $\Delta = 0$ can be realized in two ways. First is the above case, when $8AB \pm iM^2 = 0$. In the second case we solve the quartic (34) using assumption $AB \mp iM^2 = 0$, and we get three distinct roots. But now we discuss the two roots case, thus $8AB \pm iM^2 = 0$ (consequently $AB \mp iM^2 \neq 0$). Now Proposition 4.1 and Lemma 4.3 ensure that the fibration has two cusps. The equations $L = \pm iM$ and $8AB \pm iM^2 = 0$ lead to $M^3 = 8ABL$.

Conversely, if the fibration has two fibers of type II then $\Delta = 0$. There are two possibilities: either the quartic has a triple root, or it has two double roots. We showed that the case of a triple root is equivalent to the appearance of a fiber of type III. Consequently the remaining possibility is the case of two double roots. Above we analyzed the case of two double roots, which led to the equations $8AB \pm iM^2 = 0$ and $L^2 = -M^2 \neq 0$.

5.2.4. *Three roots.* The quartic has three distinct roots if the discriminant (35) vanishes, but Δ_0 and Δ_1 do not. In light of the above results we have two cases. The first is described in Subsection 5.2.3, namely $L = \pm iM$ and $AB \mp iM^2 = 0$. Indeed, if we substitute $L = \pm iM$ and $AB \mp iM^2 = 0$ to Δ_0 or Δ_1, these two expressions do not vanish, but $\Delta = 0$. The second case comes from $L \neq \pm iM$ and $\Delta = 0$. There is only one case in Proposition 4.1 where the fibration has three singular points; in both cases the singular fibers are $II + 2I_1$.

In the other direction, the cusp and two fishtails imply that the quartic has three distinct roots, hence $\Delta = 0$, $\Delta_0 \neq 0$ and $\Delta_1 \neq 0$. By the previous results, these are equivalent to the above two cases.

5.2.5. *Four roots with section.* The quartic has four distinct roots if and only if $\Delta \neq 0$. The three possible cases are listed in Proposition 4.1 and we distinguish the cases based on the existence of a section and the number of singular fibers.

We compute the number of singular fibers from (33). We use notation of (3):

$$t = -\frac{1}{4z_1}\left(2B^2z_1^3 + 3BMz_1^2 + \left((\mu_- + \mu_+)^2 - 2(a_- + a_+)(b_- + b_+)\right)z_1 + AL\right). \tag{37}$$

Let us denote the four distinct roots of (34) by y_i $(i = 1, ..., 4)$. Denote the value of t by t_i after the substitution of z_1 with y_i in Equation (37). Two roots (say y_1 and y_2) provide singularities on the same curve, if and only if $t_1 = t_2$. Equivalently:

$$0 = t_1 - t_2 = -\frac{y_1 - y_2}{4y_1y_2}\left(2B^2y_2y_1^2 + 2B^2y_2^2y_1 + 3BMy_2y_1 - AL\right). \tag{38}$$

We can simplify with the factor $-\frac{y_1-y_2}{4y_1y_2}$. Similarly, we can express all $(t_i - t_j)$ factor, where $i < j$ and $i, j \in \{1, ..., 4\}$. Obviously, the four distinct roots provide three or less values for t if and only if

$$T_1 := (t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_2 - t_3)(t_2 - t_4)(t_3 - t_4) \tag{39}$$

vanishes. Plug the simplified $t_i - t_j$ factors (expression in equation (38) and similar others) to (39). The expression T_1 is a symmetric polynomial in variables y_1, y_2, y_3, y_4, hence can be rewritten as a polynomial of the elementary symmetric polynomials $\sigma_1 = y_1+y_2+y_3+y_4$, $\sigma_2 = y_1y_2 + y_1y_3 + y_1y_4 + y_2y_3 + y_2y_4 + y_3y_4$, $\sigma_3 = y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4$ and $\sigma_4 = y_1y_2y_3y_4$. We do not reproduce the expression of T_1 in terms of the σ_i here, because we prefer to switch to the parameters A, B, M and L. Indeed, by Vieta's formulas the symmetric polynomials can also be expressed by the coefficients of the quartic of (34):

$$\sigma_1 = -\frac{4B}{3A}, \quad \sigma_2 = \frac{2AL + B^2}{3A^2}, \quad \sigma_3 = 0, \quad \sigma_4 = -\frac{M^2}{3A^2}.$$

Using these formulae, a tedious calculation provides the following form of the expression of (39):

$$\begin{aligned} T_1 = -\frac{2A^5}{B}(L - M)(L + M)\big(&-256A^3B^3 + 192A^2B^2LM- \\ &-3AB\left(9L^4 - 2L^2M^2 + 9M^4\right) + 4L^3M^3\big). \end{aligned}$$

Notice that the discriminant (35) appears in the expression, hence we get

$$T_1 = -\frac{2A^5}{B}(L - M)(L + M)\Delta.$$

Now, we consider another expression, which vanishes when at least three t_i values are equal $(i = 1, 2, 3, 4)$ or $t_i = t_j$ and $t_k = t_l$ for distinct indices $(i, j, k, l \in \{1, 2, 3, 4\})$.

$$T_2 := \sum_{\substack{i,j=1 \\ i<j}}^{4} \frac{1}{t_i - t_j} \prod_{\substack{k,l=1 \\ k<l}}^{4} (t_k - t_l).$$

Expressing T_2 in terms of A, B, L, M as we did for T_1, we get

$$\begin{aligned} T_2 = -\frac{A^4}{B}\big(&256A^3B^3L + 48A^2B^2M\left(3M^2 - 7L^2\right) + \\ &+12ABL\left(9L^4 - 8L^2M^2 + 3M^4\right) - 13L^4M^3 + 9L^2M^5\big). \end{aligned}$$

Now we concentrate on the cases with section. If $\Delta \neq 0$, then $T_1 = 0$ is equivalent to $L = \pm M$. This means that if the pencil has a section (by Lemma 5.5) then Lemma 4.3 guarantees that a fiber of type I_2 occurs, hence the possibility of four fishtail fibers is excluded. Moreover, T_2 simplifies to

$$T_2 = \frac{4A^4M}{B}\left(M^2 \mp 4AB\right)^3,$$

where $\mp$ corresponds to the sign in $L = \pm M$. If $T_1 = T_2 = 0$ we have two cases.

First, $4AB = \pm M^2$ and $L = \pm M$ lead to $4AB = LM$. The latter means $\Delta_0 = 0$ and in Subsection 5.2.2 we showed then that Δ vanishes, which is excluded from the present analysis.

Second, if $M = L = 0$ then the discriminant becomes $-256A^6B^6$. This does not vanish, thus we get four singular points in two singular fibers, which is the $2I_2$ case.

Finally, if $T_1 = 0$ but $T_2 \neq 0$ then $L = M = 0$ is excluded. Moreover if $L = \pm M \neq 0$ then $T_2 \neq 0, \Delta \neq 0 \iff \Delta \neq 0, M^2 = L^2 \neq 0$ and this gives the $I_2 + 2I_1$ case.

The converse direction is obvious according to Proposition 4.1 and Lemma 4.7.

5.2.6. *Four roots without a section.* Now, if the pencil has no section (i.e. $L^2 \neq M^2$) then Lemma 4.3 provides that the fibration has no I_2-fiber, hence by Proposition 4.1 the only possibility is four fishtail fibers. We saw that four distinct t values means $T_1 \neq 0$, and then the four singular points lie in four distinct fibers.

Conversely, if the fibration has $4I_1$, then the quartic has four roots, hence $\Delta \neq 0$. Arguing indirectly, we suppose $L^2 = M^2$, then we do not have four distinct t values, i. e. four singular fibers. Hence $L^2 \neq M^2$.

Proof of Proposition 5.1. The case-analysis above exhausts all possibilities, and hence verifies Proposition 5.1. The result is conveniently summarized in Table 1. □

5.3. **The discussion of cases appearing in Proposition 5.2.** We assume that the polar part of the Higgs field is semisimple near q_1 and non-semisimple near q_2. The pencil is given by Equations (25) and (28). The base locus of the pencil consists of three points: $(0, a_-)$ and $(0, a_+)$ in the chart (z_1, w_1) and $(0, b_{-4})$ in the chart (z_2, w_2). Consequently the fibration has a singular fiber of type I_3^*. The other possible singular fibers are listed in Proposition 4.1.

If the polar part of the Higgs field is non-semisimple near q_1 and semisimple near q_2, we get a very similar case. We only have to replace Equations (25) with (27) and Equations (28) with (26).

The notations of polynomials will be the same as Subsection 5.2 with different values, but this will not lead to confusion. We express the coefficients p_i and q_j from Equations (25) and (28), $i = 0, 1, 2$ and $j = 0, 1, 3, 4$. (Note that we also use the equation $p_2^2 + 4q_4 = 0$.) The characteristic polynomial (17) in both trivializations:

$$\begin{aligned}\chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 + \left(2b_{-4}z_1^2 + b_{-2}z_1 - (a_- + a_+)\right) w_1 + b_{-4}^2 z_1^4 + \\ &+ (b_{-4}b_{-2} - b_{-3})\, z_1^3 - t z_1^2 + (a_+\lambda_- + a_-\lambda_+)\, z_1 + a_-a_+, \\ \chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + \left((a_- + a_+)\, z_2^2 - b_{-2}z_2 - 2b_{-4}\right) w_2 + a_-a_+z_2^4 \\ &+ (a_+\lambda_- + a_-\lambda_+)\, z_2^3 - t z_2^2 + (b_{-4}b_{-2} - b_{-3})\, z_2 + b_{-4}^2.\end{aligned} \tag{40}$$

Similarly, we consider the partial derivatives (30) of $\chi_{\vartheta_1}(z_1, w_1, t)$ to determine the singular points. We express w_1 and t from the second and the third equations and simplify using Condition (24b):

$$\begin{aligned} w_1(z_1) =& \frac{1}{2}\left(-2b_{-4}z_1^2 - b_{-2}z_1 + a_- + a_+\right), \\ t(z_1) =& \frac{1}{4z_1}\left(-6b_{-3}z_1^2 + \left(4a_-b_{-4} + 4a_+b_{-4} - b_{-2}^2\right) z_1 - (a_- - a_+)(\lambda_-\lambda_+)\right).\end{aligned} \tag{41}$$

Substitute these into the Equation (30a) and get

$$0 = 2b_{-3}z_1^3 + \left((a_- + a_+)\, b_{-2} + 2a_+\lambda_- + 2a_-\lambda_+\right) z_1 - (a_- - a_+)^2.$$

Rewrite the equation with the notation in (3) and use Condition (24b) to get

$$0 = 2b_{-3}z_1^3 - ALz_1 - A^2. \tag{42}$$

This is a cubic polynomial, generally it has three distinct roots, and this corresponds to the fact that there are at most three singular fibers in the fibration.

The cubic polynomial of (42) with variable z_1 has multiple roots if and only if its discriminant vanishes. Consider the discriminant

$$\Delta = 4A^3 b_{-3}\left(2L^3 - 27Ab_{-3}\right).$$

Assumption 1.1 and a similar statement as Lemma 5.6 implies $A \neq 0$. The expression $\Delta_0 = 6Ab_{-3}L$ is related to the fact whether the cubic polynomial has double or triple roots.

We note that if we use the matrices (21) and (22) then the discriminant will be

$$4B^3 a_{-3}\left(2M^3 - 27Ba_{-3}\right).$$

5.3.1. *One root.* The cubic equation of (42) has one root if and only if $\Delta = \Delta_0 = 0$, which is equivalent to $b_{-3} = 0$ in our case. The cubic of (42) therefore reduces to a linear equation

$$0 = -ALz_1 - A^2.$$

If $b_{-3} = 0$ then the pencil $\chi_{\vartheta_2}(z_2, w_2, t)$ in Equation (40) becomes the same as the pencil $\chi_{\vartheta_2}(z_2, w_2, t)$ in the semisimple case in Subsection 5.2 with the assumption $B = 0$. Indeed, choose the parameters $b_- = b_+ = b_{-4}$, $\mu_+ = 0$ and $\mu_- = b_{-2}$ in Equation (32). Using condition (24b) we get Equation (40) with $b_{-3} = 0$. Consequently Lemma 5.6 applies and shows that the pencil has no smooth curves and the resulting fibration is not elliptic. Notice that the tangents of the curves of the pencil of (40) at $(0, b_{-2})$ are $-\frac{2}{b_{-2}}$.

5.3.2. *Two roots.* The cubic has two roots iff $\Delta = 0$ and $\Delta_0 \neq 0$. The latter inequality does not give constraints, because $A \neq 0$, $b_{-3} \neq 0$ and if $L = 0$ this leads to $b_{-3} = 0$ that is a contradiction. Thus $\Delta = 0$ provides

$$2L^3 = 27Ab_{-3}.$$

Compute the cubic's roots with this restriction, arriving to two distinct roots: a single root $\frac{3A}{L}$ and a double root $-\frac{3A}{2L}$. According to Proposition 4.1, the fibration has a cusp and a fishtail fibers.

Conversely, if the fibration has a fiber of type II and an I_1, then the cubic must have two roots. This happens exactly when $\Delta = 0$.

5.3.3. *Three roots.* Finally, if $\Delta \neq 0$ the cubic has three distinct roots and it gives rise to three singular fibers which are (by Proposition 4.1) all fishtails. The converse direction is also trivial.

Proof of Proposition 5.2. We conclude that the singular fibers next to the type I_3^* fiber are the following

- three I_1-fibers iff $\Delta \neq 0$, and
- a type II fiber and an I_1-fiber iff $\Delta = 0$,

hence we verified Proposition 5.2. □

5.4. **The discussion of cases appearing in Proposition 5.3.** The polar part of the Higgs field is non-semisimple near both q_i ($i = 1, 2$). Equations (26) and (28) determine the spectral curve and the pencil. The base locus of the pencil consists of two points: $(0, a_{-4})$ in the chart (z_1, w_1) and $(0, b_{-4})$ in the chart (z_2, w_2). The fiber at infinity is of type I_4^*.

Again, express the coefficients p_i and q_j from equations (26) and (28), $i = 0, 1, 2$ and $j = 0, 1, 3, 4$. The characteristic polynomial (17) has the following shape:

$$\begin{aligned}\chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 + \left(2b_{-4}z_1^2 - a_{-2}z_1 - 2a_{-4}\right)w_1 + b_{-4}^2 z_1^4 + \\ &+ \left(b_{-4}b_{-2} - b_{-3}\right)z_1^3 - tz_1^2 + \left(a_{-4}a_{-2} - a_{-3}\right)z_1 + a_{-4}^2.\end{aligned}$$

Using Condition (24d), we solve the second and third equations from (30) to determine the singular points:

$$
\begin{aligned}
w_1(z_1) &= \frac{1}{2}\left(-2b_{-4}z_1^2 + a_{-2}z_1 + 2a_{-4}\right),\\
t(z_1) &= \frac{1}{4z_1}\left(-6b_{-3}z_1^2 + \left(8a_{-4}b_{-4} - a_{-2}^2\right)z_1 - 2a_{-3}\right).
\end{aligned}
\tag{43}
$$

Now, substitute w_1 and t into Equation (30a) and get

$$0 = b_{-3}z_1^3 - a_{-3}z_1. \tag{44}$$

Generally, a cubic polynomial has three distinct roots. One of the roots of (44) is $z_1 = 0$, but due to the remark before Subsection 4.1 the fiber component of the curve at infinity with multiplicity 2 has no singular point of any other curve in the pencil, hence the cubic equation can be reduced to following quadric:

$$0 = b_{-3}z_1^2 - a_{-3}. \tag{45}$$

The quadric polynomial has a double root if and only if its discriminant $\Delta = 4a_{-3}b_{-3}$ vanishes.

5.4.1. *One root.* The cubic has one root in two ways.

First, if $b_{-3} = 0$ and $a_{-3} \neq 0$, then the cubic (44) reduces to $0 = a_{-3}z_1$. The singular point lies on the line $z_1 = 0$, but this pencil does not give an elliptic fibration. If $a_{-3} = b_{-3} = 0$, then from Equation (45) we see that the pencil contains curves only of higher multiplicity, hence it does not contain a smooth curve.

If $b_{-3} \neq 0$ then the cubic (44) has one root if and only if $a_{-3} = 0$, but this leads $z_1 = 0$ again.

5.4.2. *Two roots.* If the discriminant of the quadric of (45) is nonzero, then the quadric always has two distinct roots which are never zero. In other words the fibration has two singular fibers which are fishtails due to Proposition 4.1.

Proof of Proposition 5.3. The above discussion shows that no other possibility in the case of two base points can arise, thus we proved Proposition 5.3. □

6. THE ORDER OF POLES ARE 3 AND 1

This section contains the cases of Theorems 2.4, 2.5, 2.6 and 2.7 without the parabolic weights. More precisely we will prove the following propositions. (Recall that X is given in Notation 3.17.)

Proposition 6.1. *Assume that the polar part of the Higgs field is regular semisimple near q_1 and regular semisimple near q_2. Then X is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_6$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

1. *if $L = \pm M$ and $B^2 = \pm 4AM$ (implying $\Delta = 0$), then a type III and an I_1 fibers;*
2. *if $L = \pm M$ and $B^2 = \mp 12AM$ (implying $\Delta = 0$), then a type II and an I_2 fibers;*
3. *if $L = \pm M$, $B^2 \neq \pm 4AM$ and $B^2 \neq \mp 12AM$ (and so $\Delta \neq 0$), then a type I_2 and two I_1 fibers;*
4. *if $L \neq \pm M$, $\Delta = 0$ and $B = 0$, then two type II fibers;*
5. *if $L \neq \pm M$, $\Delta = 0$ and $B \neq 0$, then a type II and two I_1 fibers;*
6. *if $L \neq \pm M$ and $\Delta \neq 0$, then four type I_1 fibers.*

where for Δ see (65).

Once again, this result can be conveniently summarized in Table 2.

	$\mathbf{L = \pm M}$		$\mathbf{L \neq \pm M}$	
$\mathbf{\Delta = 0}$	$\mathbf{B^2 = \pm 4AM}$	$III + I_1$	$\mathbf{B = 0}$	$2II$
	$\mathbf{B^2 = \mp 12AM}$	$II + I_2$	$\mathbf{B \neq 0}$	$II + 2I_1$
$\mathbf{\Delta \neq 0}$	$I_2 + 2I_1$		$4I_1$	

TABLE 2. The type of singular curves in (3,1) case with four base points. In this case the fiber at infinity is an $\tilde{E}_6$ fiber.

Proposition 6.2. *Assume that the polar part of the Higgs field is regular semisimple near q_1 and non-semisimple near q_2. Then X is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_6$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $\Delta = 0$ and $L = 0$, then a type IV fiber;
(2) if $\Delta = 0$ and $L \neq 0$, then a type II and an I_2 fibers;
(3) if $\Delta \neq 0$ and $L = 0$, then a type I_3 and an I_1 fibers;
(4) if $\Delta \neq 0$, $L \neq 0$ and $B^2 = -2AL$, then a type III and an I_1 fibers;
(5) if $\Delta \neq 0$, $L \neq 0$ and $B^2 \neq -2AL$, then a type I_2 and two I_1 fibers;

where $\Delta = 4A^2\left(B^2 - 6AL\right)$.

The table summarizing this case has the following shape:

	$\mathbf{L = 0}$	$\mathbf{L \neq 0}$	
$\mathbf{\Delta = 0}$	IV	$II + I_2$	
$\mathbf{\Delta \neq 0}$	$I_3 + I_1$	$\mathbf{B^2 = -2AL}$	$III + I_1$
		$\mathbf{B^2 \neq -2AL}$	$I_2 + 2I_1$

TABLE 3. The type of singular curves in (3,1) case with three base points. In this case the fiber at infinity is an $\tilde{E}_6$ fiber.

Proposition 6.3. *Assume that the polar part of the Higgs field is non-semisimple near q_1 and regular semisimple near q_2. Then X is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $\Delta = 0$ and $Q \neq 0$, then a type II and an I_1 fibers;
(2) if $\Delta \neq 0$ and $Q \neq 0$, then three type I_1 fibers;

where $\Delta = M^2\left(27M^2Q^2 - 4R^3\right)$. If $Q = 0$ then the fibration is not elliptic.

Proposition 6.4. *Assume that the polar part of the Higgs field is non-semisimple near q_1 and non-semisimple near q_2. Then X is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) if $R = 0$ and $Q \neq 0$, then a type III fiber;
(2) if $R \neq 0$ and $Q \neq 0$, then a type I_2 and an I_1 fibers.

If $Q = 0$ then the fibration is not elliptic.

Remark 6.5. *Again, we will prove implications in converse directions as well.*

6.1. **Local form of irregular Higgs bundles.** In this section we give the local description of the spectral curves on the Hirzebruch surface $\mathbb{F}_2$ in the case (3,1). Let U_i be suitable affine open charts near q_i on $C = \mathbb{C}P^1$. The line bundle $K_C(3\cdot\{q_1\}+\{q_2\})$ has the trivializations κ_i $(i=1,2)$.

$$\kappa_1 = \frac{\mathrm{d}z_1}{z_1^3},$$
$$\kappa_2 = \frac{\mathrm{d}z_2}{z_2}.$$

The conversion from κ_1 to κ_2 is the following:

$$\kappa_1 = \frac{\mathrm{d}z_1}{z_1^3} = -z_2\mathrm{d}z_2 = -z_2^2\kappa_2. \tag{46}$$

The trivialization κ_i induces a trivialization κ_i^2 on $K_C(3\cdot\{q_1\}+\{q_2\})^{\otimes 2}$, $i=1,2$.

We proceed as in Section 5.1. Consider the two charts V_i on Hirzebruch surface which are the complements of the section at infinity in preimages of U_i under p $(i=1,2)$ (and disjoint from the section at infinity). Let $\zeta\in\Gamma(\mathbb{F}_2, p^*K_C(D))$ be the canonical section, and introduce $w_i\in\Gamma(V_i,\mathcal{O})$ by

$$\zeta = w_i\otimes\kappa_i.$$

The conversion between w_1 to w_2 is $w_2 = -z_2^2 w_1$ again.

Consider an irregular Higgs bundle $(\mathcal{E},\theta)$ in some trivializations; the local forms of θ near q_1 or q_2 are the following

$$\theta = \sum_{n\geq -3} A_n z_1^n\otimes\mathrm{d}z_1, \text{ or } \quad \theta = \sum_{n\geq -1} B_n z_2^n\otimes\mathrm{d}z_2, \tag{47}$$

where $A_n, B_n\in\mathfrak{gl}(2,\mathbb{C})$.

Recall the definition of the characteristic polynomial $\chi_\theta(\zeta)$ from (11). Let us set $\vartheta_1 = \sum_{n\geq 0} A_{n-3}z_1^n$ or $\vartheta_2 = \sum_{n\geq 0} B_{n-1}z_2^n$ so that we have

$$\theta = \vartheta_i\otimes\kappa_i.$$

If we divide by κ_i in (11) then the characteristic polynomial may be rewritten as

$$\chi_{\vartheta_i}(w_i) = \det(w_i\mathrm{I}_{\mathcal{E}}-\vartheta_i) = w_i^2 + w_i f_i + g_i, \tag{48}$$

with locally defined functions f_i, g_i such that

$$s_1 = f_i\kappa_i, \quad s_2 = g_i\kappa_i^2.$$

Now, as $K(3\cdot\{q_1\}+\{q_2\})\cong\mathcal{O}(2)$, the coefficients f_1 and g_1 in the κ_1 trivialization are polynomials in z_1 of degree 2 and 4, respectively:

$$f_1(z_1) = -(p_2z_1^2+p_1z_1+p_0),$$
$$g_1(z_1) = -(q_4z_1^4+q_3z_1^3+q_2z_1^2+q_1z_1+q_0),$$

where all coefficients are elements of $\mathbb{C}$.

Similarly, in the κ_2 trivialization we get

$$f_2(z_2) = p_0z_2^2+p_1z_2+p_2,$$
$$g_2(z_2) = -\left(q_0z_2^4+q_1z_2^3+q_2z_2^2+q_3z_2+q_4\right).$$

Finally, from (48) we get two polynomials in variables z_1, w_1 or z_2, w_2. These polynomials are the local forms of the spectral curves in $Z_C(D)$:

$$\chi_{\vartheta_1}(z_1,w_1) = w_1^2 - \left(p_2z_1^2+p_1z_1+p_0\right)w_1 - \left(q_4z_1^4+q_3z_1^3+q_2z_1^2+q_1z_1+q_0\right), \tag{49}$$

$$\chi_{\vartheta_2}(z_2,w_2) = w_2^2 + \left(p_0z_2^2+p_1z_2+p_2\right)w_2 - \left(q_0z_2^4+q_1z_2^3+q_2z_2^2+q_3z_2+q_4\right). \tag{50}$$

The spectral curve has an expansion near q_1 and q_2 in which these parameters have a geometric meaning. The lowest index matrices A_n and B_n in (47) encode the base locus of a pencil. The matrices A_{-2} and A_{-1} encode the tangent and the second derivative of the curve of a pencil. As indicated in Subsection 2.2 the letters $(S),(N),(s),(n)$ refer to the following cases which may occur independently of each other.

Near $z_1 = 0$:

(S) the semisimple case

$$\theta = \left[\begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} z_1^{-3} + \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix} z_1^{-2} + \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} z_1^{-1} + O(1)\right] \otimes \mathrm{d}z_1, \tag{51}$$

(N) the non-semisimple case

$$\theta = \left(\begin{pmatrix} b_{-6} & 1 \\ 0 & b_{-6} \end{pmatrix} z_1^{-3} + \begin{pmatrix} 0 & 0 \\ b_{-5} & b_{-4} \end{pmatrix} z_1^{-2} + \begin{pmatrix} 0 & 0 \\ b_{-3} & b_{-2} \end{pmatrix} z_1^{-1} + O(1)\right) \otimes \mathrm{d}z_1. \tag{52}$$

Near $z_2 = 0$:

(s) the semisimple case

$$\theta = \left[\begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix} z_2^{-1} + O(1)\right] \otimes \mathrm{d}z_2, \tag{53}$$

(n) the non-semisimple case

$$\theta = \left(\begin{pmatrix} b_{-1} & 1 \\ 0 & b_{-1} \end{pmatrix} z_2^{-1} + O(1)\right) \otimes \mathrm{d}z_2. \tag{54}$$

We again observe

$$\operatorname{tr}\operatorname{Res}_{z_1=0}\theta + \operatorname{tr}\operatorname{Res}_{z_1=\infty}\theta = 0,$$

which, according to the various cases, implies

$$\lambda_+ + \lambda_- + \mu_+ + \mu_- = 0, \tag{55a}$$
$$\lambda_+ + \lambda_- + 2b_{-1} = 0, \tag{55b}$$
$$b_{-2} + \mu_+ + \mu_- = 0, \tag{55c}$$
$$b_{-2} + 2b_{-1} = 0. \tag{55d}$$

The roots of the characteristic polynomial (48) in w_i have Puiseux expansions with respect to z_i. The first several terms of these expansions are equal to the eigenvalues of the matrices (51) – (54). In concrete terms:

(S) The polar part of θ near q_1 has semisimple leading-order term. The power series of the 'negative' root of $\chi_{\vartheta_1}(z_1, w_1)$ in (49) up to second order is equal to $a_- + b_- z_1 + \lambda_- z_1^2$ and the 'positive' root up to second order is equal to $a_+ + b_+ z_1 + \lambda_+ z_1^2$. We get the following equations:

$$a_- = \frac{1}{2}\left(p_0 - \sqrt{p_0^2 + 4q_0}\right), \tag{56a}$$

$$b_- = \frac{1}{2}\left(p_1 - \frac{p_0 p_1 + 2q_1}{\sqrt{p_0^2 + 4q_0}}\right), \tag{56b}$$

$$\lambda_- = \frac{1}{2}\left(p_2 - \frac{2p_0^2 q_2 + p_0\left(4p_2 q_0 - 2p_1 q_1\right) + 2p_1^2 q_0 + p_2 p_0^3 - 2q_1^2 + 8q_0 q_2}{\left(p_0^2 + 4q_0\right)^{3/2}}\right), \tag{56c}$$

$$a_+ = \frac{1}{2}\left(\sqrt{p_0^2 + 4q_0} + p_0\right), \tag{56d}$$

$$b_+ = \frac{1}{2}\left(\frac{p_0 p_1 + 2q_1}{\sqrt{p_0^2 + 4q_0}} + p_1\right), \tag{56e}$$

$$\lambda_+ = \frac{1}{2}\left(\frac{2p_0^2 q_2 + p_0\left(4p_2 q_0 - 2p_1 q_1\right) + 2p_1^2 q_0 + p_2 p_0^3 - 2q_1^2 + 8q_0 q_2}{\left(p_0^2 + 4q_0\right)^{3/2}} + p_2\right). \tag{56f}$$

(N) The polar part of θ near q_1 has non-semisimple leading-order term. This means that the polynomial $\chi_{\vartheta_1}(z_1, w_1)$ has one ramified root w_1 with branch point $z_1 = 0$. This leads to the formula $p_0^2 + 4q_0 = 0$. Simplify the roots of $\chi_{\vartheta_1}(z_1, w_1)$ using this condition. The series of two roots up to second order are equal to the series of eigenvalues of the matrix (52) up to second order. Notice that the expansions allow half integer powers of the variable z_1. The corresponding terms are:

$$b_{-6} = \frac{p_0}{2}, \tag{57a}$$

$$\sqrt{b_{-5}} = \sqrt{\frac{p_0 p_1}{2} + q_1}, \tag{57b}$$

$$\frac{b_{-4}}{2} = \frac{p_1}{2}, \tag{57c}$$

$$\frac{b_{-4}^2 + 4b_{-3}}{8\sqrt{b_{-5}}} = \frac{p_1^2 + 2p_0 p_2 + 4q_2}{4\sqrt{2p_0 p_1 + 4q_1}}, \tag{57d}$$

$$\frac{b_{-2}}{2} = \frac{p_2}{2}. \tag{57e}$$

(s) The polar part of θ near q_2 has semisimple leading-order term. The zero order terms of the series of the roots of $\chi_{\vartheta_2}(z_2, w_2)$ in (50) are equal to μ_- and μ_+:

$$\mu_- = -\frac{1}{2}\sqrt{p_2^2 + 4q_4} - \frac{p_2}{2}, \tag{58a}$$

$$\mu_+ = \frac{1}{2}\sqrt{p_2^2 + 4q_4} - \frac{p_2}{2}. \tag{58b}$$

(n) The polar part of θ near q_2 has non-semisimple leading-order term, implying that the polynomial $\chi_{\vartheta_2}(z_2, w_2)$ has one ramified root w_2 with branch point $z_2 = 0$. This leads to the formula $p_2^2 + 4q_4 = 0$. Use this equation to simplify the roots of $\chi_{\vartheta_2}(z_2, w_2)$. The zero order terms of the series of the roots of $\chi_{\vartheta_2}(z_2, w_2)$ are equal to the eigenvalues of the matrix in (54), that is

$$b_{-1} = -\frac{p_2}{2}. \tag{59}$$

As in Section 5, fix the polar part of θ. Again, the choices of the semisimple or the non-semisimple cases near q_1 and near q_2 are independent. We have four possibility for fixing the polar part, namely: $(Ss), (Sn), (Ns), (Nn)$.

It turns out that Equations (56)–(59) do not depend on q_3 (the coefficient of $g_1(z_1)$ and $g_2(z_2)$); thus we set

$$t = q_3. \tag{60}$$

The above equations corresponding to the pairs of letters determine the coefficients of s_1 and s_2 in any possibly choice. The given complex parameters (i. e. $a_\pm$, $b_\pm$ etc.) define the pencil of spectral curve of $(\mathcal{E}, \theta)$ parametrized by t, namely, the characteristic polynomial $\chi_\theta(\zeta)$.

According to the introduction in Section 4, the pencil χ_θ gives rise to an elliptic fibration on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ with some singular fibers. Let us denote the spectral curves by $\chi_{\vartheta_1}(z_1, w_1, t)$ in the κ_1 trivialization, and by $\chi_{\vartheta_2}(z_2, w_2, t)$ in the κ_2 trivialization.

According to the remark before Subsection 4.1, the fiber component of the curve at infinity with multiplicity 3 intersects every curve in the pencil in its smooth points. Therefore it suffices to consider the κ_2 trivialization, i. e. the chart (z_2, w_2). In identifying the singular fibers in the pencil, we look for triples (z_2, w_2, t) such that (z_2, w_2) fits the curve with parameter t, and the partial derivatives below vanish:

$$\chi_{\vartheta_2}(z_2, w_2, t) = 0, \tag{61a}$$

$$\frac{\partial \chi_{\vartheta_2}(z_2, w_2, t)}{\partial w_2} = 0, \tag{61b}$$

$$\frac{\partial \chi_{\vartheta_2}(z_2, w_2, t)}{\partial z_2} = 0. \tag{61c}$$

If no triple (z_2, w_2, t) lies on the fiber component of the curve at infinity with multiplicity 1 then the triples are in one-to-one correspondence with the singular points in the singular fibers (for the same reasons after Equations (30) in Subsection 5.1). If such a triple lies on the fiber component of the curve at infinity with multiplicity 1 then we will have to blow up this point and we will have to compute some singular points on a new chart.

Before the discussion of the cases, we consider a lemma about sections.

Lemma 6.6. *Consider the second Hirzebruch surface $\mathbb{F}_2$ with a pencil of type $(3,1)$ having four or three base points, two in the fiber with multiplicity three and two or one in the other fiber. The pencil contains a section if and only if*

- $L \pm M = 0$ *in the case of four base points,*
- $L = 0$ *in the case of three base points.*

Proof. The proof is very similar to the proof of Lemma 5.5. The two sections in the κ_1 trivialization are

$$\theta_{\pm,1} = (a_\pm + b_\pm z_1 + \lambda_\pm z_1^2 + \dots)\kappa_1.$$

If the pencil has four base points, the sections in the κ_2 trivialization are: $\theta_{\pm,2} = (\mu_\pm + \dots)\kappa_2$. If the pencil has three base points, the sections in the κ_2 trivialization are: $\theta_{\pm,2} = (b_{-1} + \dots)\kappa_2$. Four base points give the same two equations as in Lemma 5.5. Three base points give the following:

$$b_{-1} = -\lambda_+,$$
$$b_{-1} = -\lambda_-.$$

Equation (55b) ensures that both rows are fulfilled simultaneously and give $L = 0$.

The converse is also true, namely, if $L = 0$ then above two equations hold and the sections $\theta_\pm$ satisfy these two conditions. □

6.2. **The discussion of cases appearing in Proposition 6.1.** The polar part of the Higgs field is semisimple near both q_i $(i = 1, 2)$. Namely, the spectral curve Z_t and the pencil (specified by this spectral curve, together with the curve at infinity) are determined by Equations (56) and (58). The base locus of the pencil consists of four points: $(0, a_-)$ and $(0, a_+)$ in the chart (z_1, w_1) (on the fiber with multiplicity three) and $(0, \mu_-)$ and $(0, \mu_+)$ in the chart (z_2, w_2). The fibration has a singular fiber of type $\widetilde{E}_6$.

Express the coefficients p_i and q_j from the equations listed in cases (S) and (s), $i = 0, 1, 2$ and $j = 0, 1, 2, 4$. The characteristic polynomial (48) in the κ_2 trivialization becomes

$$\begin{aligned}\chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + \left((a_- + a_+) z_2^2 + (b_- + b_+) z_2 + \lambda_- + \lambda_+\right) w_2 + a_- a_+ z_2^4 + \\ &+ (a_+ b_- + a_- b_+) z_2^3 + (a_+ \lambda_- + a_- \lambda_+ + b_- b_+) z_2^2 - t z_2 + \mu_- \mu_+.\end{aligned}$$

Recall that if the polar part is semisimple near both poles then the fiber component of the curve at infinity with multiplicity 1 has no singular point. Hence the equations of (61) determine the singular points. Express w_2 and t from the second and the third equations by z_2.

$$
\begin{aligned}
w_2(z_2) &= -\frac{1}{2}\left((a_- + a_+) z_2^2 + (b_- + b_+) z_2 + \lambda_- + \lambda_+\right), \\
t(z_2) &= -\frac{1}{2}\left(2(a_- - a_+)^2 z_2^3 + 3(a_- - a_+)(b_- - b_+) z_2^2 + \right.\\
&\quad \left. + \left(2(a_- - a_+)(\lambda_- - \lambda_+) + (b_- - b_+)^2\right) z_2 + (b_- + b_+)(\lambda_- + \lambda_+)\right).
\end{aligned} \tag{62}
$$

We will need the t value later, so we simplify (62) using the notation of (4):

$$
t(z_2) = -A^2 z_2^3 - \frac{3}{2} AB z_2^2 - \frac{1}{2}\left(2AL + B^2\right) z_2 - \frac{1}{2}(b_- + b_+)(\lambda_- + \lambda_+). \tag{63}
$$

Now, substitute the resulting expressions into the Equation (61a) and get

$$
\begin{aligned}
0 =& 3(a_- - a_+)^2 z_2^4 + 4(a_- - a_+)(b_- - b_+) z_2^3 + \\
&+ \left(2(a_- - a_+)(\lambda_- - \lambda_+) + (b_- - b_+)^2\right) z_2^2 - (\lambda_- + \lambda_+)^2 + 4\mu_-\mu_+.
\end{aligned}
$$

Rewrite the equation with the notation in (4) with Condition (55a):

$$
0 = 3A^2 z_2^4 + 4AB z_2^3 + (2AL + B^2) z_2^2 - M^2. \tag{64}
$$

The roots of the quartic give the z_1 values of the corresponding singular points in the singular curves. The roots are in one-to-one correspondence with singular points in the semisimple case. Since (64) is a quartic polynomial, generally it has four distinct roots, and this corresponds to the fact that there are at most four singular fibers in the fibration.

The quartic polynomial of (64) with variable z_2 has multiple roots if and only if its discriminant

$$
-16A^2M^2 \cdot \left(48A^4\left(L^2 + 3M^2\right)^2 + 64A^3B^2L\left(L^2 - 9M^2\right) + 24A^2B^4\left(L^2 + 3M^2\right) - B^8\right)
$$

vanishes.

Remark 6.7. *A similar calculation as in Lemma 5.6 gives that the cases $a_- = a_+$ and $\mu_- = \mu_+$ lead to the non-regular semisimple case and do not give elliptic fibrations.*

Since $A \neq 0$ and $M \neq 0$ by Assumption 1.1, we can simplify the above expression by $-16A^2M^2$ and set

$$
\Delta = 48A^4\left(L^2 + 3M^2\right)^2 + 64A^3B^2L\left(L^2 - 9M^2\right) + 24A^2B^4\left(L^2 + 3M^2\right) - B^8. \tag{65}
$$

We will see that the further expressions below are connected to the fact whether the quartic has double, triple or quadruple roots.

$$
\begin{aligned}
\Delta_0 &= \left(2AL + B^2\right)^2 - 36A^2M^2, \\
\Delta_1 &= -48A^3\left(12A^3\left(L^2 + 3M^2\right) - 20A^2B^2L - 13AB^4 + 4B^3\right).
\end{aligned}
$$

6.2.1. *One root.* The fibration has one root in the chart (z_2, w_2) if and only if $\Delta = \Delta_0 = \Delta_1 = 0$. If we solve these three equations for any three variables (e. g. B, L, M or A, B, L) then we get $A = 0$ or $M = 0$ or we do not get solution. Hence one root in the semisimple case is not possible.

6.2.2. *Two roots (one triple root).* The pencil has two singular points and according to Proposition 4.2 there are two possible cases, namely type $III + I_1$ and type $2II$.

On the other hand, the quartic has two roots in two different cases.

In the first case, the quartic has a triple root. This is equivalent to $\Delta = \Delta_0 = 0$ and $\Delta_1 \neq 0$. Start with $\Delta_0 = 0$:

$$
\left(2AL + B^2\right)^2 = 36A^2M^2.
$$

Express B^2, then Δ may be rewritten as

$$
\begin{aligned}
B^2 &= -2AL \pm 6AM, \\
\Delta &= 1728A^4M^2(L \mp M)^2.
\end{aligned}
$$

Since $A \neq 0 \neq M$ but $\Delta = 0$ we get $L = \pm M$. This implies $B^2 = \pm 4AM$.

Substitute B and L into Δ_1 with the assumption $\Delta_0 = 0$, and get $\Delta_1 = 16A^2M^2$. This implies that Δ_1 never vanishes.

Since $L \pm M = 0$, Lemma 6.6 implies that the pencil contains a section. Due to Lemma 4.7 only a fiber of type III can appear in the fibration.

For the converse direction, we suppose the pencil has a singular fibers of type III and an I_1 fiber. Obviously the pencil has two singular points and $\Delta = 0$. Lemma 4.3 provides that the pencil has a section, consequently $L = \pm M$. The discriminant becomes $-\left(B^2 \mp 4AL\right)^3\left(12AL \pm B^2\right)$, which vanishes if and only if $B^2 = \mp 12AL$ or $B^2 = \pm 4AL$. In the first case substitute B^2 and $L = \pm M$ to Δ_0 and Δ_1. Since none of these vanish, the fibration has three distinct roots and that is a contradiction. The second case verifies the part of the proposition.

6.2.3. *Two roots (two double roots).* In the second case, the quartic has two double roots, hence the shape of the quartic polynomial is

$$0 = c_1(z - c_2)^2(z - c_3)^2. \tag{66}$$

Expand the equation and denote the coefficients by r_0, r_1, r_2, r_3, r_4 in ascending order. There are two relations among the coefficients:

$$\begin{aligned} 64r_4^3r_0 &= \left(r_3^2 - 4r_2r_4\right)^2, \\ r_3^3 + 8r_1r_4^2 &= 4r_2r_3r_4. \end{aligned}$$

Replace the r_i's by the coefficients of the quartic 64 and get

$$\begin{aligned} 0 &= 108A^2M^2 + \left(B^2 - 6AL\right)^2, \\ 0 &= 6A^2BL - AB^3. \end{aligned}$$

Solve these equations for B and L with the assumptions $A \neq 0$ and $M \neq 0$, and get

$$\begin{aligned} B &= 0, \\ L &= \pm i\sqrt{3}M. \end{aligned}$$

We see that the pencil has no section (due to $L \pm M \neq 0$). Proposition 4.2 and Lemma 4.3 provide that the fibration has two cusps.

In the converse direction, if the fibration has two fibers of type II then the quartic has two double roots because triple root leads to a fiber of type III as shown above. The fact that the quartic has two double roots implies Equation (66), implying $B = 0$ and $L = \pm i\sqrt{3}M$. The conditions $B = 0$ and $L = \pm i\sqrt{3}M$ are equivalent to $B = 0$ and $\Delta = 0$, the condition listed in the proposition.

6.2.4. *Three roots with section.* The quartic has three roots if and only if $\Delta = 0$ but Δ_0 and Δ_1 do not vanish. According to Proposition 4.2 we have two cases again, which can be distinguished by the existence of a section.

First, we suppose the pencil has a section, i. e. $L = \pm M$. The discriminant becomes

$$0 = -\left(B^2 \mp 4AM\right)^3\left(B^2 \pm 12AM\right). \tag{67}$$

We already met the $B^2 = \pm 4AM$ case, hence the one possible case is $B^2 = \mp 12AM$. Above we computed Δ_0 and Δ_1 in this case and they do not vanish. The appearance of an I_2 fiber is the consequence of Lemma 4.7. Proposition 4.2 provides that the fibration contains an additional cusp fiber.

In the reverse direction $\Delta = 0$ and $L = \pm M$ follows immediately. We excluded the $B^2 = \pm 4AM$ case, thus the former equality leads to $B^2 = \mp 12AM$.

6.2.5. *Three roots without section.* Again $\Delta = 0$, $\Delta_0 \neq 0$, $\Delta_1 \neq 0$ but we suppose the pencil has no section. We will be using the process of elimination.

In Subsection 6.2.3 we saw that the $\Delta = 0$, $\Delta_0 \neq 0$, $\Delta_1 = 0$ case is equivalent to $\Delta = 0$, $B = 0$ and $L \neq \pm M$. In Subsection 6.2.2 we deduced that $\Delta = 0$, $\Delta_0 = 0$, $\Delta_1 \neq 0$ implies the existence of a section. Hence if $L \neq \pm M$ then $\Delta = 0$, $\Delta_0 \neq 0$, $\Delta_1 \neq 0$ is equivalent to $\Delta = 0$, $B \neq 0$. Proposition 4.2 and Lemma 4.3 guarantee that the pencil has a cusp and two fishtail fibers next to the $\widetilde{E}_6$ fiber.

Conversely, if the pencil has a II and two I_1 fibers, then $\Delta = 0$ and the pencil contains no section. Since $B = 0$ leads to the case of two cusps, we conclude that $B \neq 0$.

6.2.6. *Four roots.* The fibration has four singular points if and only if $\Delta \neq 0$. The characterization of singular fibers in Proposition 4.2 give three possible cases which differ by the number of singular curves in the pencil: $4I_1, I_2 + 2I_1, I_3 + I_1$.

Let us denote the four singular distinct roots of (64) by y_i ($i = 1, ..., 4$). Denote t by t_i after the substitution of z_2 with y_i in Equation (63). Two roots (say y_1 and y_2) provide singularities on the same curve if and only if $t_1 = t_2$. Equivalently:

$$\begin{aligned} 0 =& t_1 - t_2 = \\ =& (y_1 - y_2)\left(2A^2y_1^2 + 2A^2y_2^2 + 2A^2y_1y_2 + 3ABy_1 + 3ABy_2 + 2AL + B^2\right). \end{aligned} \tag{68}$$

We can simplify with $(y_1 - y_2)$. Similarly, we can express all $(t_i - t_j)$ factor, where $i < j$ and $i, j \in \{1, ..., 4\}$. Obviously, the four distinct roots provide three or less values for t if and only if

$$T_1 := (t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_2 - t_3)(t_2 - t_4)(t_3 - t_4) \tag{69}$$

vanishes. Plugging the simplified $t_i - t_j$ factors (expression in Equation (68) and similar others) to (69). The expression T_1 became a symmetric polynomial in variables y_1, y_2, y_3, y_4, hence can be written as a polynomial of the elementary symmetric polynomials as in case $(2, 2)$ in Subsection 5.2.5.

The vanishing condition $T_1 = 0$ would yield a long expression, but $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ can be determined from the coefficients of polynomial (64) by Vieta's formulas. The relations between the symmetric polynomials and the coefficients are the following:

$$\begin{aligned} \sigma_1 =& -\frac{4B}{3A}, \\ \sigma_2 =& \frac{2AL + B^2}{3A^2}, \\ \sigma_3 =& 0, \\ \sigma_4 =& -\frac{M^2}{3A^2}. \end{aligned}$$

This transforms Equation (69) to

$$\begin{aligned} T_1 =& \frac{4}{729}A^2(L - M)(L + M)\Big(48A^4\left(L^2 + 3M^2\right)^2 + \\ &+ 64A^3B^2L\left(L^2 - 9M^2\right) + 24A^2B^4\left(L^2 + 3M^2\right) - B^8\Big) = \\ =& \frac{4}{729}A^2(L - M)(L + M)\Delta \end{aligned}$$

Consider a similar expression

$$T_2 := \sum_{\substack{i,j=1 \\ i<j}}^{4} \frac{1}{t_i - t_j} \prod_{\substack{k,l=1 \\ k<l}}^{4} (t_k - t_l).$$

This expression vanishes exactly when three t_i values are equal ($i = 1, ...4$) or $t_i = t_j$ and $t_k = t_l$ for distinct indices ($i, j, k, l \in \{1, 2, 3, 4\}$). Executing on T_2 the same simplification as on T_1, we get

$$\begin{aligned}T_2 =&\frac{8}{729}\left(192A^5\left(L^5+3L^3M^2\right)+16A^4B^2\left(13L^4-72L^2M^2+27M^4\right)+\right.\\&\left.+12A^3B^4L\left(5L^2+3M^2\right)-AB^8L\right).\end{aligned}$$

It is easy to conclude that $T_1 \neq 0$ and $\Delta \neq 0$ is equivalent to $L \neq \pm M$. By Lemma 6.6 the latter condition means that the pencil has no section. By Lemma 4.3 the fiber of type I_2 or I_3 is excluded, hence the only possibility is four fishtail fibers.

Consider the case when the pencil has a section (that is, $L = \pm M$). $T_1 = 0$ immediately follows, and

$$T_2 = \pm\frac{8}{729}AM\left(4AM \mp B^2\right)^3\left(12AM \pm B^2\right).$$

The condition $T_1 = 0$ (and $\Delta \neq 0$) ensures that the pencil has no four distinct singular fibers, hence the $4I_1$ case is not possible. On the other hand, if $T_1 = T_2 = 0$ we exactly get back the $III + I_1$ and $II + I_2$ cases, therefore the I_3 fiber cannot appear in the fibration. $T_2 \neq 0$ (and $L = \pm M$ holds) is equivalent to $B^2 \neq \pm 4AM$ and $B^2 \neq \mp 12AM$, and the latter is equivalent to $\Delta \neq 0$ (see Equation (67)). $T_1 = 0$ gives rise to three distinct t values. This means the fibration has an I_2 fiber and two I_1's.

By Proposition 4.2 and Lemma 4.7 the converse is also obvious.

Proof of Proposition 6.1. It is not hard to see that the cases discussed in the subsection above exhaust all possibilities, and hence verify Proposition 6.1. For the summary of the results, see Table 2. □

6.3. **The discussion of cases appearing in Proposition 6.2.** The polar part of the Higgs field is semisimple near q_1 and non-semisimple near q_2. Equations (56) and (59) determine the spectral curve and the pencil. The base locus consists of three points: $(0, a_-)$ and $(0, a_+)$ in the chart (z_1, w_1) and $(0, b_{-1})$ in the chart (z_2, w_2). Consequently, the fibration has a singular fiber of type $\widetilde{E}_6$.

Again we express p_i and q_j from Equations (56) and (59) ($i = 0, 1, 2$ and $j = 0, 1, 2, 4$). The characteristic polynomial (48) in the κ_2 trivialization become the following:

$$\begin{aligned}\chi_{\vartheta_2}(z_2, w_2, t) =&w_2^2+\left(\left(a_-+a_+\right)z_2^2+\left(b_-+b_+\right)z_2+\lambda_-+\lambda_+\right)w_2+a_-a_+z_2^4+\\&+\left(a_+b_-+a_-b_+\right)z_2^3+\left(a_+\lambda_-+a_-\lambda_++b_-b_+\right)z_2^2-tz_2+b_{-1}^2.\end{aligned}\tag{70}$$

As before, we identify the partial derivatives (61) to determine the singular points. Not necessarily all of them, because some singular points can come from the blow-up. We express w_2 and t from the second and the third equations and simplify using Condition (55b):

$$\begin{aligned}w_2(z_2) =&-\frac{1}{2}\left(\left(a_-+a_+\right)z_2^2+\left(b_-+b_+\right)z_2+\lambda_-+\lambda_+\right),\\ t(z_2) =&-\frac{1}{2}\left(2\left(a_--a_+\right)^2z_2^3+3\left(a_--a_+\right)\left(b_--b_+\right)z_2^2+\right.\\&\left.+\left(2\left(a_--a_+\right)\left(\lambda_--\lambda_+\right)+\left(b_--b_+\right)^2\right)z_2+\left(b_-+b_+\right)\left(\lambda_-+\lambda_+\right)\right).\end{aligned}$$

We will need the t value later, so we simplify that using the notation of (4):

$$t(z_2) = -\frac{1}{2}\left(2A^3z_2^3+3ABz_2^2+\left(2AL+B^2\right)z_2+\left(b_-+b_+\right)\left(\lambda_-+\lambda_+\right)\right).\tag{71}$$

Now, substitute w_2 and t into Equation (61a) and get

$$\begin{aligned}0 =&3\left(a_--a_+\right)^2z_2^4+4\left(a_--a_+\right)\left(b_--b_+\right)z_2^3+\\&+\left(2\left(a_--a_+\right)\left(\lambda_--\lambda_+\right)+\left(b_--b_+\right)^2\right)z_2^2+4b_{-1}^2-\left(\lambda_-+\lambda_+\right)^2.\end{aligned}$$

Reformulate the equation with the notation of (4) and use Condition (55b):

$$0 = 3A^2z_2^4+4ABz_2^3+\left(2AL+B^2\right)z_2^2.\tag{72}$$

A quartic polynomial generally has four distinct roots, but now (72) has two $z_2 = 0$ roots, thus it has at most three distinct roots. Moreover, by Lemma 4.5 we will blow up the point $z_2 = 0, w_2 = b_{-1}$ and compute singular points in a new chart. The number of singular points may eventually be four.

The quartic Equation (72) can be reduced by z_2^2 and we get a quadric polynomial:

$$0 = 3A^2 z_2^2 + 4ABz_2 + \left(2AL + B^2\right). \tag{73}$$

The discriminant of the quadric is

$$\Delta = 4A^2\left(B^2 - 6AL\right), \tag{74}$$

where $A \neq 0$ by Assumption 1.1 (see Remark 6.7).

6.3.1. *Blow-up on the chart* (z_2, w_2). The root $z_2 = 0$ of the quartic creates the possibility of existence of some possible singular points. According to Lemma 4.5, the fibration may contain a fiber with a component which appears in the blow-up only. For this reason we would like to compute the blow-up of χ_{ϑ_2} in Equation (70). (See the lower diagram in the Figure 8.) The appropriate blow-up procedure is the following. First we blow up the $(z_2 = 0, w_2 = b_{-1})$ point in the chart (z_2, w_2). The exceptional divisor is

$$E_1 = \{z_2 = 0, w_2 = b_{-1}, [\alpha : \beta]\}.$$

Choose the local chart given by $\alpha \neq 0$; the variables in χ_{ϑ_2} should be replaced as follows: $z_2 = \alpha$, $w_2 = \alpha\beta$. Next we blow-up the origin $(\alpha = 0, \beta = 0)$. The exceptional divisor is

$$E_2 = \{\alpha = 0, \beta = 0, [u : v]\}.$$

Choose the local chart given by $u \neq 0$; the variables in χ_{ϑ_2} should be replaced as follows: $\alpha = uv$, $\beta = u$. The u axis will be part of the possible singular fiber. Let us denote by χ_b the pencil χ_{ϑ_2} in the chart $u \neq 0$. Reduce the expression by Condition (55b), and divide by the exceptional divisor (which is $u^2 v$), and get

$$\begin{aligned}\chi_b(u,v,t) =& a_- a_+ u^6 v^3 + (a_- + a_+)\, u^3 v^2 + (a_+ b_- + a_- b_+)\, u^4 v^2 + \\ &+ \left((a_- - a_+)\, b_{-1} + a_-\lambda_+ - a_+\lambda_+ + b_- b_+\right) u^2 v + (b_- + b_+)\, uv + v + \\ &+ (b_- + b_+)\, b_{-1} - t.\end{aligned}$$

Let us compute the singular points of this family of curves on the chart (u, v). As before, we search the triples (u, v, t) such that the point (u, v) lies on the curve with parameter t, and the partial derivatives vanish (as in Equation (61)). The second and third equations do not contain the variable t, thus we solve them in the variable u and v, and we get four roots (hence four is the maximal number of singular points in this case). Two of these lie on the u axis:

$$u = -\frac{2}{(b_- + b_+) \pm \sqrt{2AL + B^2}}, v = 0. \tag{75}$$

The corresponding t values are

$$(b_- + b_+)\, b_{-1} \tag{76}$$

in both cases. This means that the possible two singular points on the u axis lie on the same curve.

6.3.2. *One root.* Since the point $(z_2 = 0, w_2 = b_{-1})$ is always the preimage of a singular point in $\mathbb{F}_2$, the necessary condition of the appearance of the fiber of type IV is that there is no other singular point in the chart (z_2, w_2). This requires that the quadric (73) has one $z_2 = 0$ root, thus $\Delta = 0$ and the linear term vanishes. These are equivalent to $L = B = 0$, and also equivalent to $\Delta = L = 0$.

For the other direction we need the blow-up of a point $(0, b_{-1})$ in the chart (z_2, w_2) with the condition $L = B = 0$. Previously we computed the singular points in the blow-up. By substituting $L = B = 0$ to (75), it turns out that the pencil has one singular point:

$u = -\frac{1}{b_+}, v = 0$. Proposition 4.2 implies that the fibration contains a singular fiber of type IV.

6.3.3. *Two roots (one triple root).* If the quartic has one root in the chart (z_2, w_2) then there are no two singular points in the blowing up. Thus one of the roots is not on $z_2 = 0$. There are two possible cases.

The first case is when $(0, b_{-1})$ with any t values is a singular point with multiplicity three, i. e. $z_2 = 0$ is a triple root of the quartic (72). The second case will be the case of two double roots in the next subsection. In the case of triple root, the quadric (73) has no constant term (i. e. $2AL + B^2 = 0$) and its discriminant does not vanish. The quadric and the discriminant (74) become

$$\begin{aligned} 0 =& Az_2 \left(3Az_2 + 4B\right), \\ 0 \neq& -32A^3 L. \end{aligned} \tag{77}$$

Obviously $L = 0$ leads to the case of one root, hence we can assume now that $L \neq 0$.

We need to determine how many singular points come from the point $(0, b_{-1})$ during the blow-up. Instead of computing on a chart in the blow-up, we consider the tangents of the curves of the pencil of (70) at the point $(0, b_{-1})$. There are three cases

- If the tangent, as function $z_2(w_2)$, is zero (i. e. the tangent consists of the w_2 axis), then the curve is a smooth curve in the pencil or a smooth point of a singular fiber.
- If the computation leads two different tangents for one curve, then the curve has a fishtail singularity at the point $(0, b_{-1})$.
- If the tangent is unique and it is not zero, then the curve has a cusp or higher order singularity at the point $(0, b_{-1})$.

We saw in Subsection 6.3.1 that the $z_2 = 0$ root of the quartic of (72) leads to the singular points on the u-axis on the chart (u, v). Now, $z_2 = 0$ is a triple root and it means that the u-axis has at most three distinct singular points. Suppose that the u-axis has exactly three distinct singular points. Since the u-axis is part of a singular fiber (see Lemma 4.5), we get a fiber with three singular points. The only possible case is an I_3 fiber. In this case, according to Lemma 4.4 the pencil contains a section. Then Lemma 6.6 guarantees $L = 0$, contradicting our assumption $L \neq 0$.

Hence the u-axis has at most two singular points. Let us compute the tangents of the curves of the pencil of (70). In Subsection 6.3.1 it was shown that the singular points in the u-axis fit on the same fiber with t value given by (76). Substitute Equation (76) to the equation of the pencil χ_{ϑ_2} and use condition (55b). Denote the curve by Z_{sing} in the κ_2 trivialization:

$$\begin{aligned} Z_{\text{sing}}(z_2, w_2) =& w_2^2 + \big(\left(a_- + a_+\right) z_2^2 + \left(b_- + b_+\right) z_2 + \lambda_- + \lambda_+\big) w_2 + a_- a_+ z_2^4 + \\ &+ \left(a_+ b_- + a_- b_+\right) z_2^3 + \left(a_+ \lambda_- + a_- \lambda_+ + b_- b_+\right) z_2^2 + \\ &+ \frac{1}{2}\left(b_- + b_+\right)\left(\lambda_- + \lambda_+\right) z_2 + \frac{1}{4}\left(\lambda_- + \lambda_+\right). \end{aligned}$$

Solve the equation $Z_{\text{sing}} = 0$ in variable w_2

$$\begin{aligned} w_2^{(1,2)} =& \frac{1}{2}\big(\left(a_- + a_+\right) z_2^2 - \left(b_- + b_+\right) z_2 - \lambda_- - \lambda_+ \pm \\ &\pm \sqrt{\left(\left(a_- - a_+\right) z_2 + b_- - b_+\right)^2 + 2\left(a_- - a_+\right)\left(\lambda_- - \lambda_+\right) z_2}\big). \end{aligned}$$

Taking the derivatives of $w_2^{(1,2)}$ with respect to the variable z_2 at $z_2 = 0$ and using the notation of (4) we get:

$$-\frac{1}{2}\left(b_- + b_+\right) \pm \sqrt{2AL + B^2}.$$

If $2AL + B^2 = 0$ (when the quartic (72) has a triple root) then the two derivatives are equal and the tangent of the curve Z_{sing} is unique. It leads to the case when the curve Z_{sing} has a cusp singularity at the point $(0, b_{-1})$ and the pencil has an unique singular point on the u-axis in the blow-up. We note that the slope of the tangent of the function $z_2(w_2)$ is not zero, because $b_- + b_+ \neq \infty$.

In conclusion, the pencil has altogether two singular points: one of them comes from the blow-up of the point $(0, b_{-1})$, the other one comes from the root $z_2 = -\frac{4B}{3A}$ of the quadric (77). By Propositions 4.2 and Lemma 4.5 the only possible case is that the fibration has a type III and a type I_1 singular fibers.

Conversely, we suppose that the fibration has a type III and an I_1 fibers. Due to Subsection 6.3.2 two singular points come from two roots of the quartic. As at the beginning of Subsection 6.3.3, we have two cases but only the first case gives two singular points, thus $B^2 + 2AL = 0$ and $L \neq 0$.

6.3.4. *Two roots (two double roots).* If the quartic has a double root in a point $z_2 \neq 0$, then the discriminant (74) of the quadric vanishes:

$$0 = B^2 - 6AL.$$

If $L = B = 0$, we could get the fiber IV. Thus $L \neq 0$, consequently $B \neq 0$. The blow-up of the point $(0, b_{-1})$ is

$$u = -\frac{2}{(b_- + b_+) \pm \sqrt{8AL}}, v = 0.$$

These points never coincide, hence the pencil has three singular points. Moreover, the latter two points lie on the same curve, hence the pencil has altogether two singular curves. Proposition 4.2 leads to one possible case: the fibration has a type II and an I_2 fibers.

Suppose now that the fibration has a type II and an I_2 singular fibers. Then we have three singular points and three possible cases which produce these. In the first and second cases the quartic has two roots, as we discussed above. We saw that if $z_2 = 0$ is a triple root, then the fibration has a type III fiber. If a point $z_2 \neq 0$ is a double root, we get a cusp and an I_2 fibers. This happens exactly when the discriminant vanishes and $L \neq 0$. In the third case the quartic has three distinct roots, so that the blow-up of the point $(0, b_{-1})$ must contain one singular point. This condition means that the two points in (75) coincide. This happens exactly when $0 = 2AL + B^2$, which leads to a triple root and fiber of type III.

6.3.5. *Three roots with section.* The quartic has three distinct roots if and only if the discriminant (74) does not vanish, and $B^2 \neq -2AL$. If $B^2 \neq -2AL$, then the two singular points in the blow-up are distinct. Thus three roots give four singular points. Proposition 4.2 offers three possibilities: $4I_1, I_3 + I_1, I_2 + 2I_1$. The two singular points which came from the blow-up lie on the same fiber. Hence the case of four fishtails is not possible.

We will separate the remaining cases by the existence of section. If the pencil has a section, then Lemma 6.6 provides $L = 0$. The roots of the quadric (73) and the roots which came from the blow-up are

$$-\frac{B}{A}, -\frac{B}{3A}, -\frac{1}{b_+}, -\frac{1}{b_-}.$$

We have $B \neq 0$, since $\Delta \neq 0$ and $L = 0$ (i. e. $B = 0$ leads to type IV fiber), so these roots are distinct. It is enough to show that the fibration has two singular fibers only. Substitute the two roots of the quadric to the expression of (71):

$$t_1 := -\frac{1}{2}(b_- + b_+)(\lambda_- + \lambda_+),$$
$$t_2 := \frac{1}{2}\left(\frac{2B^3}{27A} - (b_- + b_+)(\lambda_- + \lambda_+)\right).$$

Otherwise, as we have seen in Equation (76), the t value which appeared in the blow-up was

$$t_3 := (b_- + b_+)\, b_{-1}.$$

Condition (55b) gives $-2b_{-1} = \lambda_+ + \lambda_-$ and this makes t_1 and t_3 equal. The result gives a singular fiber with three singular points. The only possible case is an I_3 fiber. Finally, due to Proposition 4.2, the other singular fiber is I_1.

Assume that there is an I_3 and a I_1 singular fibers in the fibration. $\Delta \neq 0$ is obvious, and Lemma 4.4 provides that there is a section, and furthermore $L = 0$.

6.3.6. *Three roots without section.* If the pencil has no section then $L \neq 0$. The equations give four distinct roots as above and at least two singular fibers. Proposition 4.2 and the process of elimination ensure that we will get the case $I_2 + 2I_1$, because the appearance of an I_3 fiber leads to $L = 0$.

Conversely, suppose that the fibration has an I_2 fiber and two fishtail fibers. Four singular points require $\Delta \neq 0$. The $L \neq 0$ condition comes from the fact that the fibration has no I_3 or IV fiber. Finally, if $B^2 = -2AL$ then all of this leads to $III + I_1$ case, so $B^2 \neq -2AL$.

Proof of Proposition 6.2. Once again, the case-analysis above discussed all possible cases, providing the proof of the claim given in Proposition 6.2. □

6.4. **The discussion of cases appearing in Proposition 6.3.** The polar part of the Higgs field is non-semisimple near q_1 and semisimple near q_2. Equations (57) and (58) give the spectral curve and the pencil. The base locus consists of three points: $(0, b_{-6})$ in the chart (z_1, w_1) (on the fiber with multiplicity three) and $(0, \mu_-)$ and $(0, \mu_+)$ in the chart (z_2, w_2). The fibration has a singular fiber of type $\widetilde{E}_7$.

We will need the characteristic polynomial (48) in both trivializations:

$$\begin{aligned}\chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 - \left(b_{-2} z_1^2 + b_{-4} z_1 + 2b_{-6}\right) w_1 + \mu_- \mu_+ z_1^4 - t z_1^3 + \\ &+ \left(b_{-6} b_{-2} - b_{-3}\right) z_1^2 + \left(b_{-6} b_{-4} - b_{-5}\right) z_1 + b_{-6}^2, \\ \chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + \left(2b_{-6} z_2^2 + b_{-4} z_2 + b_{-2}\right) w_2 + b_{-6}^2 z_2^4 + \left(b_{-6} b_{-4} - b_{-5}\right) z_2^3 + \\ &+ \left(b_{-6} b_{-2} - b_{-3}\right) z_2^2 - t z_2 + \mu_- \mu_+ .\end{aligned} \tag{78}$$

It is enough to analyze the fibration in κ_2 trivialization, but later we need to use the equation of the pencil in κ_1 trivialization.

We solve the equations of (61) using Condition (55c). We express w_2 and t from the second and the third equations and substitute them into the first.

$$\begin{aligned} w_2(z_2) =& -\frac{1}{2}\left(2b_{-6} z_2^2 + b_{-4} z_2 + b_{-2}\right), \\ t(z_2) =& -\frac{1}{2}\left(6b_{-5} z_2^2 + \left(b_{-4}^2 + 4b_{-3}\right) z_2 + b_{-4} b_{-2}\right), \\ 0 =& 8b_{-5} z_2^3 + \left(b_{-4}^2 + 4b_{-3}\right) z_2^2 - b_{-2}^2 + 4\mu_- \mu_+ .\end{aligned}$$

We can rewrite the last expression using the notation in (4) and again use the residuum condition (55c) to get

$$0 = Q z_2^3 + R z_2^2 - M^2. \tag{79}$$

The roots of this cubic give the z_2 values of singular points in the singular fibers. Now, the roots are in one-to-one correspondence with singular points, because the fiber component of the curve at infinity with multiplicity 1 has two base points. Cubic polynomials generally have three distinct roots, and this corresponds to the fact that there are at most three singular fibers in the elliptic fibration.

The discriminant of the cubic of (79) is

$$\Delta = M^2\left(27 M^2 Q^2 - 4R^3\right).$$

$M = 0$ leads to the non-regular semisimple case (see Remark 6.7), and according to Assumption 1.1 we have $M \neq 0$ in the following.

One more expression characterizes the number of the roots:

$$\Delta_0 = R^2.$$

6.4.1. *One root.* The cubic polynomial (79) has one root if and only if the discriminant Δ and Δ_0 vanish. This leads to $R = Q = 0$, furthermore $M = 0$, which is contradiction.

6.4.2. *Two roots.* If $Q = 0$, the polynomial of (79) becomes quadratic, and hence it has two roots.

Lemma 6.8. *The equation* $0 = Rz_2^2 - M^2$ *does not give an elliptic fibration.*

Proof. The $Q = 0$ condition is equivalent to $b_{-5} = 0$. Consider the tangents of the curves of the pencil of (78) in $(z_1 = 0, w_1 = b_{-6})$. Compute the implicit derivative of $\chi_{\vartheta_1}(z_1, w_1)$ in the points $(0, b_{-6})$ with the condition $b_{-5} = 0$ and get

$$\frac{\frac{\partial \chi_{\vartheta_1}}{\partial w_1}}{\frac{\partial \chi_{\vartheta_1}}{\partial z_1}} = -\frac{2}{b_{-4}}.$$

Since $b_{-4} \neq \infty$, the curves intersect the $z_1 = 0$ axis transversally in $(0, b_{-6})$. This is a singular point and $(0, b_{-6})$ lies on all curves in the pencil. The pencil has no smooth curves, hence the resulting fibration is not elliptic. (See remark before Subsection 4.1.) □

If $Q \neq 0$ and $\Delta = 0$, the cubic polynomial has two distinct roots, one double root which gives a cusp fiber and a single root which gives a fishtail fiber; by Proposition 4.2 and Lemma 4.6 there is no other possibility.

Conversely, the existence of a fiber of type II and I_1 implies $\Delta = 0$.

6.4.3. *Three roots.* Finally, if $\Delta \neq 0$ and $Q \neq 0$ the cubic has three distinct roots. Then by Proposition 4.2 and Lemma 4.6 all singular fibers are I_1. The converse direction is obvious.

Proof of Proposition 6.3. The cases enlisted above now prove Proposition 6.3: to have an elliptic fibration we need $Q \neq 0$, and the fiber at infinity is $\widetilde{E}_7$, while the further fibers are

- if $\Delta \neq 0$, then three I_1-fibers, and
- if $\Delta = 0$, then a type II fiber and an I_1 fiber.

□

6.5. **The discussion of cases appearing in Proposition 6.4.** The polar part of the Higgs field is non-semisimple near both q_i ($i = 1, 2$). Equations (57) and (59) determine the spectral curve and the pencil. The base locus of the pencil consists of only two points: $(0, b_{-6})$ in the chart (z_1, w_1) and $(0, b_{-1})$ in the chart (z_2, w_2). Consequently the fibration has a singular fiber of type $\widetilde{E}_7$.

We will need the characteristic polynomial (48) in both trivialization:

$$\begin{aligned}\chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 + \left(-b_{-2}z_1^2 - b_{-4}z_1 - 2b_{-6}\right) w_1 + b_{-1}^2 z_1^4 - t z_1^3 + \\ &+ \left(b_{-6}b_{-2} - b_{-3}\right) z_1^2 + \left(b_{-6}b_{-4} - b_{-5}\right) z_1 + b_{-6}^2,\end{aligned} \tag{80}$$

$$\begin{aligned}\chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + \left(2b_{-6}z_2^2 + b_{-4}z_2 + b_{-2}\right) w_2 + b_{-6}^2 z_2^4 + \left(b_{-6}b_{-4} - b_{-5}\right) z_2^3 + \\ &+ \left(b_{-6}b_{-2} - b_{-3}\right) z_2^2 - t z_2 + b_{-1}^2.\end{aligned} \tag{81}$$

It is enough to analyze the fibration in κ_2 trivialization.

Again consider the equations of (61) to identify singular points; once again some singular points can come from the blowing up.

Express w_2 and t from the second and the third equations by z_2:

$$w_2(z_2) = -\frac{1}{2}\left(2b_{-6}z_2^2 + b_{-4}z_2 + b_{-2}\right),$$
$$t(z_2) = -\frac{1}{2}\left(6b_{-5}z_2^2 + \left(b_{-4}^2 + 4b_{-3}\right)z_2 + b_{-4}b_{-2}\right).$$

Substitute all these into the equation of (30a):

$$0 = 8b_{-5}z_2^3 + \left(b_{-4}^2 + 4b_{-3}\right)z_2^2 - b_{-2}^2 + 4b_{-1}^2.$$

Rewrite this polynomial with the notation in (4) and use Condition (55d):

$$0 = Qz_2^3 + Rz_2^2. \tag{82}$$

This polynomial has at most two roots: $z_2 = 0$ and $z_2 = -\frac{R}{Q}$. $z_2 = 0$ is a double root and it lies on the fiber component of the curve at infinity with multiplicity 1, thus we will need to blow up this point, similarly to Subsection 6.3 (see Lemma 4.5).

6.5.1. *Blow-up on the chart* (z_2, w_2). The $z_2 = 0$ root of the cubic provides the possibility of the existence of some possible singular points. In the same way as in Subsection 6.3.1, we compute the blow-up of χ_{ϑ_2} in Equation (81). (See the lower diagram in the Figure 8.) The blow-up operation is the same as before: we blow-up the point $(z_2 = 0, w_2 = b_{-1})$ and then we blow up the new origin. The exceptional divisors are also the same as in previous case. Finally, we get a pencil in the chart $u \neq 0$, which we denote by χ_b. We simplify the expression by Condition (55d) and divide by the exceptional divisor (which is u^2v):

$$\chi_b(u,v,t) = b_{-6}^2u^6v^3 + (b_{-6}b_{-4} - b_{-5})\,u^4v^2 + 2b_{-6}u^3v^2 - b_{-3}u^2v + b_{-4}uv + v + b_{-4}b_{-1} - t.$$

Let us compute the singular points of this family of curves on the chart (u, v). As before, we search for triples (u, v, t) such that the point (u, v) lies on the curve with parameter t, and the partial derivatives vanish (as in Equation (61)). The second and third equations do not contain the variable t, thus we solve them for u and v, and we get three roots (three is the maximal number of singular points in this case). The (u, v) coordinates of the singular points are the following:

$$\left(\frac{Q}{b_{-6}R - 4b_{-5}b_{-4}}, -\frac{R\left(b_{-6}R - 4b_{-5}b_{-4}\right)^2}{Q^3}\right), \left(-\frac{2}{b_{-4} + \sqrt{R}}, 0\right), \left(\frac{b_{-4} + \sqrt{R}}{2b_{-3}}, 0\right). \tag{83}$$

The corresponding t values are $b_{-4}b_{-1}$ in second and third cases. This means that the possible two singular points on the u axis lie on the same curve, and implies that the fibration cannot have three I_1 fibers.

6.5.2. *One root.* The equation of (82) can have only one solution (which is $z_2 = 0$) in two different ways.

First we consider the case of $Q = b_{-5} = 0$. As in Lemma 6.8, we analyze the spectral curves (80) in the point $(z_1 = 0, w_1 = b_{-6})$. We take the implicit derivative of $\chi_{\vartheta_1}(z_1, w_1)$:

$$\frac{\frac{\partial \chi_{\vartheta_1}}{\partial w_1}}{\frac{\partial \chi_{\vartheta_1}}{\partial z_1}} = -\frac{2}{b_{-4}}.$$

Since $b_{-4} \neq \infty$, the curves intersect the $z_1 = 0$ axis transversally in $(0, b_{-6})$. This is a singular point and it lies on all curves in the pencil. The pencil has no smooth curve, hence it is not an elliptic fibration in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$. (See remark before Subsection 4.1.)

Now, we consider the $Q \neq 0$ case. The polynomial (82) has a triple root $z_2 = 0$ if and only if $R = 0$. Substitute $R = 0$ in the (u, v) coordinates of the singular points (83) on the blow-up and use the notation $R = b_{-4}^2 + 4b_{-3}$. As a result, we get one singular point:

$$(u, v, t) = \left(-\frac{2}{b_{-4}}, 0, b_{-4}b_{-1}\right).$$

By the classification in Proposition 4.2, the unique singular point belongs to a fiber of type III.

Conversely, if we have a type III fiber, then the cubic of (82) has one root. This implies that $R = 0$.

6.5.3. *Two roots.* The cubic of (82) has two distinct roots if and only if $R \neq 0$ (since $Q \neq 0$ holds). As we have shown above, the fibration has at most three singular points in the chart (u, v). The singular point which comes from cubic's root $z_2 = -\frac{R}{Q}$ never coincides with the other singular point which comes from the root $z_2 = 0$. Hence we only analyze the second and third singular points from (83). Solve the following system of equations in variables R and b_{-3}:

$$\begin{aligned} -\frac{2}{b_{-4} + \sqrt{R}} &= \frac{b_{-4} + \sqrt{R}}{2b_{-3}}, \\ R &= b_{-4}^2 + 4b_{-3}. \end{aligned}$$

It has only one solution, where $R = 0$. This is a contradiction, hence the second and the third singular points do not coincide. The fibration has three distinct singular points on two singular fibers because the last two singular points lie on the fiber component which came from the blow-up (see Lemma 4.5). According to Proposition 4.2 these are an I_2 and an I_1 fibers.

In the converse direction, we suppose that the fibration has an I_2 and an I_1 fibers. Then the cubic of (82) must have two distinct roots because the blow-up operation brings up two singular points from one root. This implies $R \neq 0$.

Proof of Proposition 6.4. Once again, the cases analyzed above provide a complete proof of Proposition 6.4, showing that (since the pencil is assumed to have smooth curves) we have $Q \neq 0$ and the fibers next to the $\widetilde{E}_7$ fiber are

- if $R \neq 0$, then an I_2 and an I_1 fiber, and
- if $R = 0$, then a type III fiber.

□

7. Sheaves on curves of type I_1 and II

In Propositions 5.1, 5.2, 5.3, 6.1, 6.2, 6.3 and 6.4 we have separated cases according to the multiplicities of fibers of the Hirzebruch surface appearing in the pencil and determined the various possibilities for the remaining singular fibers of the fibration. From now on, we will study torsion-free rank-1 sheaves on the various fibers of the fibration. Therefore, by virtue of Theorem 3.16, it will be more convenient to consider the various singular curves that occur, and study torsion-free rank-1 sheaves on each one of them.

In this section, we will study torsion-free rank-1 sheaves on curves of type I_1 and II. Although the classifications of torsion-free sheaves on curves of these types are well-known [1, 3], we reproduce them in this section for sake of completeness. This analysis will finish the proof of parts 2, 3, 4 and 7 of Theorem 2.1, Theorems 2.2 and 2.3, parts 5, 6 and 4 of Theorem 2.4 and Theorem 2.6. Indeed, Propositions 5.1, 5.2, 5.3, 6.1 and 6.3 respectively show that in these cases all singular fibers of the elliptic surface are of type I_1 or II, so the proofs of the above parts of the theorems follows from a simple application of Theorem 3.16 and the results of this section.

7.1. Sheaves on curves of type I_1.

Lemma 7.1. *Assume that X_t is any curve of type I_1 and let $\delta \in \mathbb{Z}$ be given. (Actually, we will only use that X_t is a fishtail curve, not its self-intersection number.) Then isomorphism classes of invertible sheaves of degree δ on X_t are parameterized by $\mathbb{C}^\times$, and isomorphism classes of non-invertible torsion-free sheaves of rank 1 and degree δ are parameterized by* *a point. In particular, in case the moduli space is known to be a smooth elliptic surface the corresponding Hitchin fiber is of type* I_1.

Proof. We will be sketchy, as we will see similar ideas in the proofs of Lemmas 8.4 and 8.6.

The first statement follows using the cohomology long exact sequence induced by the short exact sequence of sheaves

$$0 \to \mathcal{O}^{\times}_{X_t} \to \mathcal{O}^{\times}_{\tilde{X}_t} \to \mathbb{C}^{\times} \to 0,$$

where $\tilde{X}_t$ stands for the normalization of X_t.

The length of $\mathcal{O}_{\tilde{X}_t}$ as an $\mathcal{O}_{X_t}$-module is

$$l(\mathcal{O}_{\tilde{X}_t}) = 1.$$

Any torsion-free sheaf of rank 1 on X_t is either invertible or the direct image of an invertible sheaf on $\tilde{X}_t$. The second statement then follows since $\tilde{X}_t$ is of genus 0.

For the third statement, simply notice that by Kodaira's classification, I_1 is the only degeneration of an elliptic curve in the class of $\mathbf{L}$. □

7.2. **Sheaves on curves of type** II.

Lemma 7.2. *Assume that* X_t *is any curve of type* II *(more precisely, any cuspidal rational curve) and let* $\delta \in \mathbb{Z}$ *be given. Then isomorphism classes of invertible sheaves of degree* δ *on* X_t *are parameterized by* $\mathbb{C}$*, and isomorphism classes of non-invertible torsion-free sheaves of rank* 1 *and degree* δ *are parameterized by a point. In particular, in case the moduli space is known to be a smooth elliptic surface the corresponding Hitchin fiber is of type* II.

Proof. Let $\tilde{X}_t$ stand for the normalization of X_t. We claim that there is a short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{O}^{\times}_{X_t} \to \mathcal{O}^{\times}_{\tilde{X}_t} \to \mathbb{C} \to 0.$$

Indeed, in suitable coordinates on affine charts we have

$$X_t = \operatorname{Spec}(\mathbb{C}[x,y]/(x^3 - y^2)), \quad \tilde{X}_t = \operatorname{Spec}\mathbb{C}[t]$$

and the normalization morphism $\tilde{p}$ is induced by

$$t \mapsto (t^2, t^3).$$

The cokernel of this morphism is the $\mathbb{C}$-vector space spanned by the monomial t. The first statement then follows from the induced cohomology long exact sequence.

Again, a torsion-free sheaf of rank 1 on X_t is either invertible or the direct image of an invertible sheaf on $\tilde{X}_t$. This implies the second statement because $g(\tilde{X}_t) = 0$.

Finally, by Kodaira's classification, II is the only degeneration of an elliptic curve in the class of $\mathbf{L} + \mathbf{1}$. □

8. Sheaves on curves of type III, non-degenerate case

In this section we will study torsion-free rank-1 sheaves on curves of type III and prove part (1) of Theorem 2.1 and part (1) of Theorem 2.4. Along the way, we will prove the assertion of Theorem 1.3 in the cases where the hyperplane in the weight space is induced by the existence of two irreducible components of a fiber X_t of type III.

Indeed, by Propositions 5.1 and 6.1, in these cases the elliptic fibration has a singular fiber X_t of type III (and possibly an I_1 fiber). In particular, in these cases we have parabolic weights α_i^j for $i \in \{\pm\}$ and $j \in \{q_1, q_2\}$. According to the proof of Propositions 5.1 and 6.1, all components of the type III curve cover simply the base $\mathbb{C}P^1$ (i.e. none of them lie in some fiber of p).

We will denote the components of X_t by X_+ and X_-. Up to a permutation of $X_\pm$ we see that there are two essentially different cases for the intersection of the exceptional divisors of X and the components of X_t:

(1) either X_+ intersects E_+^1, E_+^2 and X_- intersects E_-^1, E_-^2
(2) or X_+ intersects E_+^1, E_-^2 and X_- intersects E_-^1, E_+^2.

Actually, these cases are told apart by the conditions on the parameters: according to the discussion in the proof of Lemma 5.5, (1) happens if and only if $L = -M \neq 0$ and (2) happens if and only if $L = M \neq 0$. (The statement analogous to Lemma 5.5 in the $(3,1)$ case is Lemma 6.6.) In the rest of this section we will assume that we have $L = -M \neq 0$; in the case $L = M \neq 0$ the same analysis continues to be correct up to exchanging the roles of α_+^2 and α_-^2.

8.1. **Normalization, partial normalization, length, bidegree.** We begin by introducing some notation. We let X_t be the singular fiber of type III. In suitable coordinates on an open affine set U containing its singular point, it is given by

$$X_t \cap U = \mathrm{Spec}(R)$$

with

$$\begin{aligned} R &= \mathbb{C}[x,y]/I \\ I &= ((x-y^2)(x+y^2)). \end{aligned}$$

We let $(0,0)$ stand for the only singular point of X_t, given by $x = 0 = y$. We denote by $\tilde{X}_t$ the normalization of X_t. It is easy to see that there exists a partial normalization X_t' inbetween X_t and $\tilde{X}_t$. Affine open sets of these curves may respectively be written as

$$\begin{aligned} \tilde{X}_t \cap U &= \mathrm{Spec}(\tilde{R}) & \qquad \tilde{R} &= \mathbb{C}[\tilde{x},\tilde{y}]/\tilde{I} \\ X_t' \cap U &= \mathrm{Spec}(R') & R' &= \mathbb{C}[x',y']/I', \end{aligned}$$

with

$$\begin{aligned} \tilde{I} &= ((\tilde{x}-1)(\tilde{x}+1)) \\ I' &= ((x'-y')(x'+y')). \end{aligned}$$

We then have natural morphisms

$$\tilde{X}_t \xrightarrow{\tilde{p}} X_t' \xrightarrow{p'} X_t \tag{84}$$

induced over Zariski open sets V such that $(0,0) \notin V$ by the identity and over U by

$$\begin{aligned} p'(x) &= x'y', & \qquad p'(y) &= y'; \\ \tilde{p}(x') &= \tilde{x}\tilde{y}, & \tilde{p}(y') &= \tilde{y}. \end{aligned}$$

For any torsion-free coherent sheaf $\mathcal{S}$ of $\mathcal{O}_{X_t \cap U}$-modules let $\mathcal{S}_{(0,0)}$ denote the fiber of $\mathcal{S}$ at $(0,0)$.

Definition 8.1. *If $\mathcal{S}$ satisfies*

$$\mathcal{O}_{X_t,(0,0)} \subseteq \mathcal{S}_{(0,0)}$$

*then the **length** of $\mathcal{S}$ at $(0,0)$ is defined as*

$$l(\mathcal{S}) = \dim_{\mathbb{C}}(\mathcal{S}_{(0,0)}/\mathcal{O}_{X_t,(0,0)}).$$

Example 8.2. *It may be checked that R'/R is the $\mathbb{C}$-vector space with basis x', so*

$$l(p'_*\mathcal{O}_{X_t'}) = 1.$$

Similarly, the vector space $\tilde{R}/R$ has basis $\tilde{x}, \tilde{x}\tilde{y}$, hence

$$l((p' \circ \tilde{p})_*\mathcal{O}_{\tilde{X}_t}) = 2.$$

Recall from (10) that we have denoted by p the ruling of the Hirzebruch surface $Z_{\mathbb{C}P^1}(D) = \mathbb{F}_2$. We also have the map

$$\sigma : X \to Z_{\mathbb{C}P^1}(D)$$

obtained by blowing up 8 infinitely close points, studied in detail in Section 4. The specific form of the map σ depends on the configuration that we fix (i.e., the orders of the poles and whether or not the polar parts are semi-simple), but for sake of simplicity we will lift this dependence from the notation. We may thus consider the map

$$p \circ \sigma : X \to \mathbb{C}P^1.$$

Furthermore, we will use the same notation for the restriction of this map to any subscheme X_t of X. We now assume that

$$\mathcal{E} = (p \circ \sigma)_*(\mathcal{S}),$$

with $\mathcal{S}$ a torsion-free coherent sheaf of rank 1 and degree δ over X_t. A simple argument using the Riemann–Roch formula then shows that

$$\delta = d + 2. \tag{85}$$

The curve X_t has a single singular point $(0,0)$ which is a tacnode (an A_3-singularity), and it has two reduced irreducible components X_+, X_- that are rational curves. We let $\mathcal{L}(\mathcal{S})_\pm$ denote the line bundle associated to the restriction of $\mathcal{S}$ to $X_\pm$ and we set

$$\delta_\pm(\mathcal{S}) = \deg(\mathcal{L}(\mathcal{S})_\pm).$$

Definition 8.3. *The invariant*

$$(\delta_+(\mathcal{S}), \delta_-(\mathcal{S})) \in \mathbb{Z}^2 \tag{86}$$

is called the ***bidegree*** *of* $\mathcal{S}$.

For any coherent sheaf $\mathcal{S}'$ on X'_t (or $\tilde{\mathcal{S}}$ on $\tilde{X}_t$) we define its bidegree as the bidegree of the coherent sheaf $p'_*(\mathcal{S}')$ (respectively, $(p' \circ \tilde{p})_*(\tilde{\mathcal{S}})$) on X_t.

8.2. **Algebraic description of rank 1 torsion-free sheaves.** We first give a local description.

Lemma 8.4. *Any rank-1 torsion-free sheaf $\mathcal{S}$ of regular modules on $X_t \cap U$ is isomorphic to exactly one of the three following sheaves:*

(1) $\mathcal{O}_{X_t \cap U}$
(2) $p'_*(\mathcal{O}_{X'_t \cap U})$
(3) $(p' \circ \tilde{p})_*(\mathcal{O}_{\tilde{X}_t \cap U})$.

Proof. This is a special case of a more general result for arbitrary Cohen–Macaulay modules, for details see [3, Proposition 2.2.1] and [7]. Since the scheme $X_t \cap U$ is affine, sheaves of $\mathcal{O}_{X_t \cap U}$-modules correspond to modules over R. Let $\mathcal{S}$ correspond to the R-module M. Then,

$$\tilde{M} = (M \otimes_R \tilde{R}) / \operatorname{Tor}_1^{\tilde{R}}(\tilde{R}/(\tilde{x}), M \otimes_R \tilde{R})$$

is a torsion-free $\tilde{R}$-module of rank 1. Since $\tilde{R}$ is regular, we see that (possibly up to restricting U) $\tilde{M}$ is in fact a free rank-1 $\tilde{R}$-module, so choosing a generator $m \in M$ results in an $\tilde{R}$-module isomorphism

$$\tilde{M} \cong \tilde{R}.$$

Notice that this is also an R-module isomorphism.

On the other hand, the natural R-module morphism $M \to \tilde{M}$ is a monomorphism, for its kernel is torsion and M is by assumption torsion-free. To sum up, we obtain the sequence of R-modules

$$R \subseteq M \subseteq \tilde{M} \cong \tilde{R},$$

the first morphism being $r \mapsto rm$. The cases $M = R$ and $M = \tilde{R}$ clearly imply parts (1) and (3) respectively. So, assume that

$$R \subset M \subset \tilde{R},$$

i.e. that $\dim_{\mathbb{C}}(M/R) = 1$. Up to subtracting an element of R, the generator m of the R-module M is of the form

$$m = a\tilde{x}$$

for some $a \in R \setminus \{0\}$. Now, the only relations of $\tilde{R}$ as an R-module are $\tilde{y} = y, y^2\tilde{x} = x$. The dimension condition then implies that

$$a = by$$

for some $b \in R^{\times}$. In particular, this implies that

$$x' = b^{-1}m,$$

generates M, i.e. $M = R'$.

To show that the sheaves (1)–(3) are not isomorphic to each other, simply observe that length is an invariant for modules over the local ring, and that the lengths of these sheaves are all different. This finishes the proof. □

We will now describe one by one the moduli of sheaves of the above three types having given bidegree.

Lemma 8.5. *For any fixed $(\delta_+, \delta_-) \in \mathbb{Z}^2$, there exists a unique isomorphism class of invertible sheaves $\tilde{\mathcal{S}}$ on $\tilde{X}_t$ of bidegree (δ_+, δ_-).*

Proof. Since

$$\tilde{X}_t = X_+ \coprod X_-$$

with each of $X_\pm$ isomorphic to $\mathbb{C}P^1$, the statement follows from the Grothendieck–Birkhoff theorem about line bundles of given degree on $\mathbb{C}P^1$. □

We will denote the unique sheaf of bidegree (δ_+, δ_-) provided by Lemma 8.5 by $\tilde{\mathcal{S}}_{(\delta_+,\delta_-)}$.

Lemma 8.6. *For any fixed $(\delta_+, \delta_-) \in \mathbb{Z}^2$, isomorphism classes of invertible sheaves $\mathcal{S}$ on X_t such that*

$$(\delta_+(\mathcal{S}), \delta_-(\mathcal{S})) = (\delta_+, \delta_-)$$

are parameterized by $\mathbb{C}$.

Proof. We follow [6, Lemma 4.1]. Let us use the notation $\mathcal{O}^{\times}_{\tilde{X}_t,(0,0)}$ for the stalk of the sheaf $\mathcal{O}^{\times}_{\tilde{X}_t}$ at $(0,0)$. Since $\tilde{X}_t \cap U$ has two connected components, each isomorphic to an affine line, we see that

$$\mathcal{O}^{\times}_{\tilde{X}_t,(0,0)} = \{(f_+, f_-) \in \mathbb{C}\{t_+\} \oplus \mathbb{C}\{t_-\} \mid \quad f_+(0) \neq 0 \neq f_-(0)\}.$$

Since $(0,0)$ is the only singular point, the normalization map $p' \circ \tilde{p}$ is an isomorphism over affine open sets V such that $(0,0) \notin V$. Let $\mathbb{C}_{(0,0)}$ denote the sky-scraper sheaf of abelian groups with fiber $\mathbb{C}$ placed at the singular point $(0,0)$, and similarly for $\mathbb{C}^{\times}_{(0,0)}$. Then, we have a short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{O}^{\times}_{X_t} \to (p' \circ \tilde{p})_*(\mathcal{O}^{\times}_{\tilde{X}_t}) \to \mathbb{C}^{\times}_{(0,0)} \times \mathbb{C}_{(0,0)} \to 0$$

induced by

$$(f_+, f_-) \mapsto \left(\frac{f_+(0)}{f_-(0)}, f'_+(0) - f'_-(0)\right) \in \mathbb{C}^{\times} \times \mathbb{C}$$

on the stalk at $(0,0)$ and by the identity over open sets V such that $(0,0) \notin V$. Here f_i' stands for the differential of f_i with respect to t_i. The associated long exact sequence of cohomology groups reads as

$$\begin{aligned} 0 \to & H^0(X_t, \mathcal{O}_{X_t}^\times) \to H^0(X_t, (p' \circ \tilde{p})_*(\mathcal{O}_{\tilde{X}_t}^\times)) \to \mathbb{C}^\times \times \mathbb{C} \to \\ \to & H^1(X_t, \mathcal{O}_{X_t}^\times) \to H^1(X_t, (p' \circ \tilde{p})_*(\mathcal{O}_{\tilde{X}_t}^\times)) \to 0. \end{aligned} \tag{87}$$

We have

$$H^0(X_t, \mathcal{O}_{X_t}^\times) = \mathbb{C}^\times$$
$$H^0(X_t, (p' \circ \tilde{p})_*(\mathcal{O}_{\tilde{X}_t}^\times)) = \mathbb{C}^\times \times \mathbb{C}^\times,$$

and the map between them is the diagonal embedding

$$\epsilon \in \mathbb{C}^\times \mapsto (\epsilon, \epsilon) \in \mathbb{C}^\times \times \mathbb{C}^\times,$$

with cokernel given by

$$(\epsilon_+, \epsilon_-) \mapsto \frac{\epsilon_+}{\epsilon_-} \in \mathbb{C}^\times.$$

Thus, taking into account that $p' \circ \tilde{p}$ is finite, the cohomology long exact sequence simplifies into

$$0 \to \mathbb{C} \to H^1(X_t, \mathcal{O}_{X_t}^\times) \to H^1(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}^\times) \to 0.$$

As $H^1(X_t, \mathcal{O}_{X_t}^\times)$ parameterizes isomorphism classes of invertible sheaves on X_t, the lemma follows from Lemma 8.5. □

Given $\lambda \in \mathbb{C}$, we will denote the corresponding sheaf of bidegree (δ_+, δ_-) constructed in Lemma 8.6 by $\mathcal{S}_{(\delta_+,\delta_-)}(\lambda)$, and the family of sheaves of bidegree (δ_+, δ_-) will be denoted by $\mathbb{C}_{(\delta_+,\delta_-)}$.

Lemma 8.7. *For given $(\delta_+, \delta_-) \in \mathbb{Z}^2$, there exists a unique isomorphism class of invertible sheaves $\mathcal{S}'$ on X_t' of bidegree (δ_+, δ_-).*

Proof. This is similar to Lemma 8.6, so we only give a sketch. Since X_t' has a unique singular point, which is of type A_1, we get the long exact sequence of cohomology groups

$$\begin{aligned} 0 \to & H^0(X_t', \mathcal{O}_{X_t'}^\times) \to H^0(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}^\times) \to \mathbb{C}^\times \to \\ \to & H^1(X_t', \mathcal{O}_{X_t'}^\times) \to H^1(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}^\times) \to 0. \end{aligned}$$

We infer that the morphism

$$H^1(X_t', \mathcal{O}_{X_t'}^\times) \to H^1(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}^\times)$$

is an isomorphism. We conclude using Lemma 8.5. □

We will denote the unique sheaf of bidegree (δ_+, δ_-) provided by Lemma 8.7 by $\mathcal{S}'_{(\delta_+,\delta_-)}$.

8.3. **Limits of vector bundles.** In this section we will study limits of the direct images with respect to p of the sheaves introduced in Lemma 8.6. Given $(\delta_+, \delta_-) \in \mathbb{Z}^2$, let $\mathbb{C}P^1_{(\delta_+,\delta_-)}$ be the compactification of $\mathbb{C}_{(\delta_+,\delta_-)}$ by a point:

$$\mathbb{C}P^1_{(\delta_+,\delta_-)} = \operatorname{Spec}(\mathbb{C}[\lambda]) \cup \operatorname{Spec}(\mathbb{C}[\mu]), \quad \lambda^{-1} = \mu.$$

We will denote by C the base curve $\mathbb{C}P^1$.

Lemma 8.8. *Let $(\delta_+, \delta_-) \in \mathbb{Z}^2$, and assume $\delta_+ > \delta_-$. There exists a relative vector bundle $\mathcal{E}_{(\delta_+,\delta_-)}$ over*

$$C \times \mathbb{C}P^1_{(\delta_+,\delta_-)} \to \mathbb{C}P^1_{(\delta_+,\delta_-)}$$

such that

- *for any $\lambda \in \mathbb{C}_{(\delta_+,\delta_-)}$, the restriction of $\mathcal{E}_{(\delta_+,\delta_-)}$ to $C \times \{\lambda\}$ is isomorphic to*

$$(p \circ \sigma)_* \mathcal{S}_{(\delta_+,\delta_-)}(\lambda),$$

- *the restriction of $\mathcal{E}_{(\delta_+,\delta_-)}$ to $C \times \{\infty\}$ is isomorphic to $(p \circ \sigma \circ p')_* \mathcal{S}'_{(\delta_+-1,\delta_-)}$.*

Proof. We may interpret the long exact sequence (87) as follows. Let U_+, V_+ and U_-, V_- be affine coverings of X_+ and X_- respectively, so that $(p' \circ \tilde{p})^{-1}(0,0)$ consists set-theoretically of one point in each of $U_+ \cap V_+, U_- \cap V_-$. Following the notation of Lemma 8.6, we denote the corresponding points in both intersections by $t_\pm = 0$. For $i \in \{\pm\}$ let φ_i be a Čech 1-cocycle associated to a line bundle $\mathcal{L}_i$ of degree δ_i over X_i. The sheaf $\mathcal{S}_{(\delta_+,\delta_-)}(\lambda)$ arises by suitably identifying the pull-backs of the line bundles $\mathcal{L}_i$ to $\mathbb{C}[t_i]/(t_i^2)$ with each other. Namely, modifying (φ_+, φ_-) by a $\mathbb{C}^\times$-valued locally constant Čech 1-coboundary, we may arrange that $\varphi_+(0) = \varphi_-(0) = 1$. By Taylor's formula, on the vector spaces $\mathbb{C}[t_i]/(t_i^2)$ with their respective bases given by $(1, t_i)$ the coordinates of φ_i are

$$\begin{pmatrix} 1 \\ \varphi_i'(0) \end{pmatrix}.$$

Then, in order to define $\mathcal{S}_{(\delta_+,\delta_-)}(\lambda)$ we need an isomorphism

$$\mathbb{C}[t_+]/(t_+^2) \cong \mathbb{C}[t_-]/(t_-^2)$$

so that

$$\lambda = \varphi_+'(0) - \varphi_-'(0).$$

This shows that the matrix of the isomorphism between the two-dimensional vector spaces $\mathbb{C}[t_i]/(t_i^2)$ with respect to their above-mentioned bases must be

$$\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \tag{88}$$

The limit of this matrix as $\lambda \to \infty$ does not make sense immediately. However, if we apply the abelian Čech 1-coboundary $\mu \in C^1(U_+, V_+)$ to φ_+ then (88) transforms into

$$\begin{pmatrix} \mu & 0 \\ -1 & \mu \end{pmatrix}.$$

The limit

$$\lim_{\mu \to 0} \begin{pmatrix} \mu & 0 \\ -1 & \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{89}$$

is not invertible, therefore the limit

$$\lim_{\lambda \to \infty} \mathcal{S}_{(\delta_+,\delta_-)}(\lambda)$$

does not arise as the identification of any line bundles on $X_\pm$ along the subschemes given by $\mathbb{C}[t_i]/(t_i^2)$. Let us denote by M_+ the $\mathbb{C}[t_+]/(t_+)$-module $t_+\mathbb{C}[t_+]/(t_+)$ and M_- be the $\mathbb{C}[t_-]/(t_-)$-module $\mathbb{C}[t_-]/(t_-)$. Identification of X_+ with X_- along the scheme morphism $\mathbb{C}[t_+]/(t_+) \to \mathbb{C}[t_-]/(t_-)$ induced by $t_- \mapsto t_+$ produces X_t'. Formula (89) induces an identification of M_+ with M_- by

$$t_+ \mapsto -1$$

along the above scheme morphism, giving rise to an invertible sheaf on X_t'. The module M_- is the stalk at $(t_- = 0) \in X_-$ of the invertible sheaf $\mathcal{L}_- = \mathcal{O}_{X_-}(\delta_-)$. On the other hand, the module M_+ is the stalk at $(t_+ = 0) \in X_+$ of the invertible sheaf

$$\mathcal{L}_+ \otimes_{\mathcal{O}_{X_+}} \mathcal{O}_{X_+}(-1) \cong \mathcal{O}_{X_+}(\delta_+ - 1).$$

□

8.4. **The stability condition.** By assumption, we have the formula

$$0 = \deg_{\vec{\alpha}}(\mathcal{E}) = \deg(\mathcal{E}) + \alpha_+^1 + \alpha_-^1 + \alpha_+^2 + \alpha_-^2. \tag{90}$$

Set

$$\alpha_i = \alpha_i^1 + \alpha_i^2; \tag{91}$$

the introduction of these parameters is justified by the fact that stability only depends on these sums of the weights α_i^j. Because $\alpha_i^1, \alpha_i^2 \in [0,1)$ we see that for $i \in \{+,-\}$ we have

$$\alpha_i \in [0,2). \tag{92}$$

On the other hand, (90) shows that

$$\alpha_+ + \alpha_- \in \{0,1,2,3\}.$$

Now there exists a short exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{L}(\mathcal{S})_+ \oplus \mathcal{L}(\mathcal{S})_- \to \mathbb{C}_{(0,0)}^{2-l(\mathcal{S})} \to 0,$$

hence

$$\chi(\mathcal{S}) + 2 - l(\mathcal{S}) = \chi(\mathcal{L}(\mathcal{S})_+) + \chi(\mathcal{L}(\mathcal{S})_-).$$

Applying this to $\mathcal{S} = \mathcal{O}_{X_t}$ we get

$$\chi(\mathcal{O}_{X_t}) + 2 = \chi(\mathcal{O}_{X_+}) + \chi(\mathcal{O}_{X_-}).$$

Subtracting the second formula from the first we infer

$$\delta - l(\mathcal{S}) = \delta_+ + \delta_-.$$

Using this formula and (85) we can rewrite (90) as

$$0 = \delta_+ + \delta_- + l(\mathcal{S}) - 2 + \alpha_+ + \alpha_-. \tag{93}$$

Let θ be a Higgs field on $\mathcal{E}$ with spectral curve X_t. The canonical restriction morphisms

$$\mathcal{S} \to \mathcal{L}(\mathcal{S})_i$$

for $i \in \{\pm\}$ give quotient irregular parabolic Higgs bundles $(\mathcal{E}_i, \theta_i)$ of $(\mathcal{E}, \theta)$ of rank 1 and degree

$$d_i = \delta_i.$$

Furthermore, it follows that these are the only non-trivial quotient objects of $(\mathcal{E}, \theta)$. Indeed, the spectral scheme of any non-trivial quotient Higgs bundle is a subscheme of dimension 1 of the spectral curve of $(\mathcal{E}, \theta)$, flat over $\mathbb{C}P^1$, and this latter scheme has only two such non-trivial one-dimensional subschemes, that precisely correspond to the above quotient Higgs bundles. The parabolic weights at q_1 and q_2 associated to $\mathcal{E}_i$ are respectively α_i^1 and α_i^2, so the parabolic degree of $\mathcal{E}_i$ is

$$\deg_{\vec{\alpha}}(\mathcal{E}_i) = \delta_i + \alpha_i.$$

By definition, parabolic stability of $(\mathcal{E}, \theta)$ is then equivalent to the inequalities

$$0 < \delta_i + \alpha_i \tag{94}$$

for $i \in \{\pm\}$, and semi-stability is equivalent to

$$0 \le \delta_i + \alpha_i. \tag{95}$$

Condition (94) and Equation (93) immediately imply that there exist no stable Higgs bundles with spectral sheaf $\mathcal{S}$ of length 2.

8.5. **Hecke transformations.** Using the notations of Section 4 for any $i \in \{+,-\}$ and $j \in \{1,2\}$ we denote by $P_i^j \in X_i$ the intersection point $X_i \cap E_i^j$. It follows from Section 4 that P_i^j is a smooth point of X_i.

Definition 8.9. *Given a rank-*1 *torsion-free sheaf* $\mathcal{S}$ *on* X_t*, its* ***Hecke transform*** *corresponding to* i, j *is*

$$\mathcal{H}ecke_i^j(\mathcal{S}) = \mathcal{S} \otimes \mathcal{O}_{X_t}(P_i^j). \tag{96}$$

Obviously, Hecke transformations for various choices of i, j commute with each other. If $\mathcal{E} = (p \circ \sigma)_*(\mathcal{S})$ then we set

$$\mathcal{H}ecke_i^j(\mathcal{E}) = (p \circ \sigma)_*(\mathcal{H}ecke_i^j(\mathcal{S})).$$

The action of $\mathcal{H}ecke_i^j$ on the bidegree is clearly

$$(\delta_+, \delta_-) \mapsto (\delta_+ + \delta_{i+}, \delta_- + \delta_{i-})$$

with $\delta_{ii'}$ standing for the Kronecker symbol. Furthermore, we set

$$\mathcal{H}ecke_i^j(\alpha_{i'}) = \alpha_{i'} - \delta_{ii'}.$$

Lemma 8.10. *The parabolic degree of* $\mathcal{H}ecke_i^j(\mathcal{E})$ *with respect to the weights* $\mathcal{H}ecke_i^j(\alpha_i^j)$ *is the same as the parabolic degree of* $\mathcal{E}$ *with respect to the weights* (α_i^j)*. The same holds for any quotient (or sub-)object. In particular,* $\mathcal{H}ecke_i^j$ *preserves stability.*

Proof. Immediate from the definitions. □

Remark 8.11. *We do not specify the action of* $\mathcal{H}ecke_i^j$ *on the individual weights* $\alpha_{i'}^{j'}$*, only on their sums over* $j' \in \{1,2\}$ *for* i' *fixed. Again, this is justified because stability only depends on these sums, and in the sequel we will make use of this freedom of choice to make sure that the individual weights* $\mathcal{H}ecke_i^j(\alpha_{i'}^{j'})$ *all lie in the interval* $[0,1)$*. In particular, the action of* $\mathcal{H}ecke_i^j$ *on the weights is independent of* j*, so we may omit the superscript* j *from the notation of the action of Hecke transformations on* (α_i^j)*.*

For $d \in \{0,1,2,3\}$ let us introduce

$$W_d = \{\vec{\alpha} \in [0,1)^4 : \quad \alpha_+^1 + \alpha_-^1 + \alpha_+^2 + \alpha_-^2 = d\}. \tag{97}$$

Clearly, we have $W_0 = \{(0,0,0,0)\}$, and the inverse of $\mathcal{H}ecke_+$ maps this vector to

$$(\varepsilon, 1-\varepsilon, 0, 0)$$

for some $\varepsilon \in (0,1)$.

Lemma 8.12.
1. *Given* $\vec{\alpha} \in W_2$*, there exists a Hecke transformation* $\mathcal{H}ecke_i$ *such that* $\mathcal{H}ecke_i \vec{\alpha} \in W_1$*.*
2. *Given* $\vec{\alpha} \in W_3$*, there exists a composition of two Hecke transformations*

$$\mathcal{H}ecke_- \circ \mathcal{H}ecke_+$$

such that

$$\mathcal{H}ecke_- \circ \mathcal{H}ecke_+(\vec{\alpha}) \in W_1.$$

Remark 8.13. *We do not claim that these procedures are canonical, and we will see that they depend on choices.*

Proof. Assume first that we have $\alpha_+ + \alpha_- = 2$. Then at least one of $\alpha_+ \geq 1, \alpha_- \geq 1$ holds. We may assume that $\alpha_+ \geq 1$, the other case being similar. Then, there exists $\varepsilon \in (0,1)$ such that

$$\alpha_+^1 - \varepsilon \geq 0, \quad \alpha_+^2 - (1-\varepsilon) \geq 0;$$

indeed, one may pick for instance $\varepsilon = \alpha_+^1$. We may then use the Hecke transformation $\mathcal{H}ecke_+$ with the action

$$\begin{aligned}\alpha_+^1 &\mapsto \alpha_+^1 - \varepsilon \geq 0\\ \alpha_+^2 &\mapsto \alpha_+^2 - (1-\varepsilon)\\ \alpha_-^j &\mapsto \alpha_-^j.\end{aligned} \tag{98}$$

Assume now that we have $\alpha_+ + \alpha_- = 3$. Taking into account the inequalities (91) we then see that

$$\alpha_+, \alpha_- \in (1,2).$$

Then, there exists $\varepsilon \in (0,1)$ such that

$$\alpha_+^1 - \varepsilon > 0, \quad \alpha_+^2 - (1-\varepsilon) > 0.$$

We let $\mathcal{H}ecke_+$ act on $\vec{\alpha}$ by (98). Then we have $\mathcal{H}ecke_+(\vec{\alpha}) \in W_2$ and

$$\mathcal{H}ecke_+(\alpha_+^1) + \mathcal{H}ecke_+(\alpha_+^2) < 1.$$

We may then apply the first statement with $i = -$. □

8.6. **Degree -1, generic weights.** Lemma 8.12 shows that the other degree conditions can all be reduced to the analysis of the case

$$\alpha_+ + \alpha_- = 1$$

by means of suitable Hecke transformations. By assumption, we have

$$\alpha_i \in [0,2),$$

which in this case implies

$$\alpha_i \in [0,1].$$

Definition 8.14. *A weight vector $\vec{\alpha} \in W_1$ is called **generic** if*

$$\alpha_i \in (0,1) \tag{99}$$

*for both $i \in \{+,-\}$ and **special** otherwise.*

In this section, we will determine the stable and semi-stable sheaves on the singular curve X_t in the case of generic weights. Clearly, in this case stability is equivalent to semi-stability.

Assume first that $l(\mathcal{S}) = 0$, i.e. $\mathcal{S}$ is an invertible sheaf. Equation (93) reads

$$(\delta_+ + \alpha_+) + (\delta_- + \alpha_-) = 2.$$

Condition (95) then implies

$$\delta_+ = 0, \delta_- = 1 \quad \text{or} \quad \delta_+ = 1, \delta_- = 0. \tag{100}$$

Conversely, it is easy to see that under the assumption of (99) the bidegree conditions (100) imply (94). Lemma 8.6 implies that under Condition (99) stable irregular parabolic Higgs bundles with $l(\mathcal{S}) = 0$ are parameterized by

$$\mathbb{C}_{(0,1)} \coprod \mathbb{C}_{(1,0)}.$$

Let us now come to the study of sheaves with $l(\mathcal{S}) = 1$. Condition (93) then reads as

$$\delta_+ + \delta_- = 0.$$

It is easy to see that assuming (99) the only bidegree condition giving rise to stable sheaves is $\delta_+ = 0 = \delta_-$. In addition, by virtue of Lemma 8.7 this bidegree condition gives rise to a unique stable sheaf $\mathcal{S}'_{(0,0)}$.

Finally, in the case $l(\mathcal{S}) = 2$ we have already seen that there may exist no stable sheaves $\mathcal{S}$.

To sum up, we have found that for generic weight vectors $\vec{\alpha} \in W_1$ we have

$$[\mathcal{M}_t^s(\vec{\alpha})] = [\mathcal{M}_t^{ss}(\vec{\alpha})] = 2\mathbf{L} + \mathbf{1}. \tag{101}$$

As the moduli space is a complete elliptic fibration, it then follows from Kodaira's list that the fiber corresponding to the point $t \in \mathbb{C}P^1$ is of type III. This, combined with Lemma 7.1 finishes the proof of part (1) of Theorem 2.1 and part (1) of Theorem 2.4 in the case of generic weights.

Now, observe that without the restrictions $\alpha_i^j \in [0,1)$ we could have arbitrary values $\alpha_i \in \mathbb{R}$ subject to the only condition $\alpha_+ + \alpha_- = 1$. Said differently, the single quantity $\alpha_+ \in \mathbb{R}$ determines the relevant stability condition. It is clear that under these more relaxed conditions semi-stability is equivalent to stability if and only if $\alpha_+ \notin \mathbb{Z}$. Again, we call such weights ***generic***. It is straightforward to check that if $\alpha_+ \notin \mathbb{Z}$ then instead of Equation (100) stability would be equivalent to

$$\delta_+ = -\lceil \alpha_+ \rceil + 1 \quad \text{or} \quad \delta_+ = -\lceil \alpha_+ \rceil + 2, \tag{102}$$

with $\delta_- = 1 - \delta_+$. The corresponding Higgs bundles are therefore parameterized by

$$\mathbb{C}_{(-\lceil \alpha_+ \rceil + 1, \lceil \alpha_+ \rceil)} \coprod \mathbb{C}_{(-\lceil \alpha_+ \rceil + 2, \lceil \alpha_+ \rceil - 1)}. \tag{103}$$

This shows the assertion of Theorem 1.3 in the particular case of a fiber X_t of type III, hence verifies it for case (1) of Theorem 2.1 and case (1) of Theorem 2.4.

8.7. **Degree -1, special weights.** In this section we treat the case $\alpha_+ = 1, \alpha_- = 0$. Of course, the same considerations hold with + and − swapped.

Assume first that $l(\mathcal{S}) = 0$. In this case, the extensions of invertible sheaves of bidegree $\delta_+ = 1, \delta_- = 0$ are all strictly semi-stable, so the component $\mathbb{C}_{(1,0)}$ of (101) is not in the stable moduli space, hence the corresponding subset of $\mathcal{M}_t^s$ is $\mathbb{C}_{(0,1)}$. However, the above component is in the semi-stable moduli space, parameterizing strictly semi-stable Higgs bundles

$$(\mathcal{E}_{(1,0)}(\lambda), \theta_{(1,0)}(\lambda)) \tag{104}$$

for $\lambda \in \mathbb{C}$. Moreover, the bidegree condition $\delta_+ = -1, \delta_- = 2$ also leads to strictly semi-stable sheaves

$$(\mathcal{E}_{(-1,2)}(\mu), \theta_{(-1,2)}(\mu)) \tag{105}$$

for $\mu \in \mathbb{C}$. Let us set

$$(\tilde{\mathcal{E}}_{(\delta_+,\delta_-)}, \tilde{\theta}_{(\delta_+,\delta_-)}) = (p \circ \sigma \circ p' \circ \tilde{p})_* \tilde{\mathcal{S}}_{(\delta_+,\delta_-)}. \tag{106}$$

Lemma 8.15. *For any $\lambda, \mu \in \mathbb{C}$, the Higgs bundles* (104) *and* (105) *are S-equivalent to*

$$(\tilde{\mathcal{E}}_{(-1,0)}, \tilde{\theta}_{(-1,0)}).$$

Proof. The destabilizing quotient Higgs bundle for the family (105) is of degree −1, with spectral scheme X_+. By additivity of the Euler-characteristic, we see that the destabilizing Higgs subbundle of this family is then of degree 0, with spectral scheme X_-. According to (93) the associated graded Higgs bundle with respect to the Jordan–Hölder filtration of (105) corresponds to sheaves $\mathcal{S}$ on X_t with $l(\mathcal{S}) = 2$ and bidegree $(-1,0)$. We deduce from Lemma 8.5 that the associated graded Higgs bundles are isomorphic to $(p \circ \sigma \circ p' \circ \tilde{p})_* \tilde{\mathcal{S}}_{(-1,0)}$. We get the same conclusion for the family (104) along the same lines, as their destabilizing quotient Higgs bundles are of degree 0 with spectral scheme X_-. □

We infer from the lemma that in the semi-stable moduli space the subset parameterizing invertible sheaves is

$$\mathbb{C}_{(0,1)} \coprod \{(\tilde{\mathcal{E}}_{(-1,0)}, \tilde{\theta}_{(-1,0)})\}.$$

Let us now come to the case $l(\mathcal{S}) = 1$. Here, the bidegree conditions $\delta_+ = 0 = \delta_-$ and $\delta_+ = -1, \delta_- = 1$ give rise to strictly semi-stable sheaves $\mathcal{S}'_{(0,0)}$ and $\mathcal{S}'_{(-1,1)}$. Let us denote by

$$(\mathcal{E}'_{(0,0)}, \theta'_{(0,0)}), (\mathcal{E}'_{(-1,1)}, \theta'_{(-1,1)})$$

the corresponding Higgs bundles.

Lemma 8.16. *The Higgs bundles $(\mathcal{E}'_{(0,0)}, \theta'_{(0,0)})$ and $(\mathcal{E}'_{(-1,1)}, \theta'_{(-1,1)})$ are S-equivalent to*

$$(\tilde{\mathcal{E}}_{(-1,0)}, \tilde{\theta}_{(-1,0)}).$$

Proof. According to Subsection 8.4, the destabilizing quotient of $(\mathcal{E}'_{(0,0)}, \theta'_{(0,0)})$ is $(\mathcal{E}_-, \theta_-)$, with $\mathcal{E}_- \cong \mathcal{O}_{\mathbb{C}P^1}$. By additivity of the degree the corresponding destabilizing Higgs subbundle $(\mathcal{F}_-, \theta_{\mathcal{F}_-})$ must then be of degree -1 on $\mathbb{C}P^1$, so we have $\mathcal{F}_- \cong \mathcal{O}_{\mathbb{C}P^1}(-1)$. The spectral schemes of the Higgs fields θ_- and $\theta_{\mathcal{F}_-}$ are respectively X_- and X_+. Now, as both $\mathcal{E}_-$ and $\mathcal{F}_-$ are rank-1 bundles, there exists a unique Higgs field on them with given spectral scheme. The Jordan–Hölder filtration of the Higgs bundle $(\mathcal{E}_{(0,0)}, \theta)$ is thus given by

$$0 \subset (\mathcal{F}_-, \theta_{\mathcal{F}_-}) \subset (\mathcal{E}'_{(0,0)}, \theta'_{(0,0)}).$$

A similar analysis shows that the destabilizing sub- and quotient objects $(\mathcal{F}_+, \theta_{\mathcal{F}_+})$ and $(\mathcal{E}_+, \theta_+)$ of $(\mathcal{E}'_{(-1,1)}, \theta'_{(-1,1)})$ have degrees -1 and 0 respectively, and the Higgs fields on them have spectral schemes X_- and X_+ respectively. It follows from unicity of such rank-1 Higgs bundles that

$$(\mathcal{F}_-, \theta_{\mathcal{F}_-}) = (\mathcal{E}_+, \theta_+)$$
$$(\mathcal{F}_+, \theta_{\mathcal{F}_+}) = (\mathcal{E}_-, \theta_-).$$

Therefore, the graded Higgs bundles of the Jordan–Hölder filtrations of $(\mathcal{E}'_{(0,0)}, \theta'_{(0,0)})$ and $(\mathcal{E}'_{(-1,1)}, \theta'_{(-1,1)})$ agree with $(p \circ \sigma \circ p' \circ \tilde{p})_* \tilde{\mathcal{S}}_{(-1,0)}$. □

To sum up, we have found that if $\alpha_+ = 1, \alpha_- = 0$ then

$$[\mathcal{M}_t^s(\vec{\alpha})] = \mathbf{L}$$
$$[\mathcal{M}_t^{ss}(\vec{\alpha})] = \mathbf{L} + \mathbf{1}.$$

This, combined with Lemma 7.1 finishes the proof of part (1) of Theorem 2.1 and part (1) of Theorem 2.4 in the case of special weights.

8.8. **Degree -2.** We now briefly sketch how Lemma 8.12 allows us to reduce the case

$$\alpha_+ + \alpha_- = 2, \quad 0 < \alpha_+, \alpha_- < 2.$$

to the study of Subsections 8.6 and 8.7. Let us distinguish between three cases according to the values of the parabolic weights.

8.8.1. *Case* $\alpha_+ = 1 = \alpha_-$. This case gets reduced by either $\mathcal{H}ecke_+$ or $\mathcal{H}ecke_-$ to the case of degree -1 with special weights treated in Subsection 8.7. More precisely, $\mathcal{H}ecke_+$ reduces it to

$$\alpha_+ = 0, \quad \alpha_- = 1$$

while $\mathcal{H}ecke_-$ reduces it to

$$\alpha_+ = 1, \quad \alpha_- = 0.$$

8.8.2. *Case* $\alpha_+ < 1 < \alpha_-$. This case gets reduced by $\mathcal{H}ecke_-$ to the case of degree -1 with generic weights treated in Subsection 8.6. Observe that we may not apply $\mathcal{H}ecke_+$.

8.8.3. *Case* $\alpha_- < 1 < \alpha_+$. This is analogous to the previous case with + and − swapped.

To sum up, there exists a wall

$$\{\alpha_+ = \alpha_-\} \subset W_2,$$

on which $\mathcal{M}_t^{ss}(\vec{\alpha})$ is isomorphic to the moduli space of degree -1 Higgs bundles with special weights, and on the two sides of the wall $\mathcal{M}_t^{ss}(\vec{\alpha})$ is isomorphic to the moduli space of degree -1 Higgs bundles with generic weights with different bidegree conditions.

8.9. **Degree** -3**.** According to Lemma 8.12, a composition of two Hecke transformations shows that the moduli space $\mathcal{M}_t^{ss}(\vec{\alpha})$ is isomorphic to the moduli space of degree -1 Higgs bundles with generic weights.

8.10. **Degree** 0**.** A Hecke transformation shows that the moduli space $\mathcal{M}_t^{ss}(\vec{\alpha})$ is isomorphic to the moduli space of degree -1 Higgs bundles with special weights.

9. Sheaves on curves of type I_2, non-degenerate case

In this section we will study torsion-free rank-1 sheaves on curves of type I_2, and prove parts (5) and (6) of Theorem 2.1 and parts (2) and (3) of Theorem 2.4. Along the way, we will prove the assertion of Theorem 1.3 in the cases of a fiber X_t of type I_2. Indeed, by Propositions 5.1 and 6.1, in these cases the elliptic fibration has a singular fiber X_t of type I_2 (and other singular fibers that have already been studied). It follows from the proof of Propositions 5.1 and 6.1 that no component of X_t lies in any fiber of p. Again, we have parabolic weights α_i^j for $i \in \{\pm\}$ and $j \in \{q_1, q_2\}$. Much of the analysis follows the one carried out in [14] and in Section 8, so we will content ourselves with sketching the differences in the proofs as compared to Section 8, and mainly focus on the motivic wall-crossing phenomenon that has not yet been described explicitely.

Again, let us denote the components of X_t by X_+ and X_-. Just as in the beginning of Section 8, we have two combinatorial possibilities for the intersections of $X_\pm$ with the exceptional divisors E_i^j. Again, Lemma 5.5 and its counterpart Lemma 6.6 apply and give the same conditions for these possibilities in terms of the parameters as in Section 8. As in Section 8, here we will content ourselves with analyzing case (1), noting that the arguments of this section remain correct in case (2) as well, up to exchanging α_+^2 and α_-^2. However, in the current setup we need to point out the existence of a third case with I_2 fibers, not covered by the previous conditions. Namely, for $L = M = 0, \Delta \neq 0$ (part (5) of Theorem 2.1) X admits two I_2 fibers: one of them (let us temporarily call it X_{b_1}) will intersect the exceptional divisors along the scheme of point (1) and the other one (say X_{b_2}) as in point (2) from the beginning of Section 8. In this special case, we need to apply the below analysis for the fiber X_{b_1}, and redo the same analysis with α_+^2 and α_-^2 exchanged for X_{b_2}. In particular, in the degree -2 case the set of special weights for these values of the parameters is the union of the two walls defined by the equations

$$\alpha_+^1 + \alpha_+^2 = 1 \tag{107}$$

$$\alpha_+^1 + \alpha_-^2 = 1; \tag{108}$$

crossing (107) only alters the Hitchin fiber over b_1, whereas crossing (108) only alters the Hitchin fiber over b_2, in the way described later in this section.

We denote by x_1, x_2 the singular points of X_t and by

$$\tilde{p} : \tilde{X}_t \to X_t$$

the normalization of X_t. We assume that

$$\mathcal{E} = (p \circ \sigma)_*(\mathcal{S})$$

for some torsion-free sheaf $\mathcal{S}$ of $\mathcal{O}_{X_t}$-modules of rank 1. We consider the line bundles $\mathcal{L}_i$ induced by $\mathcal{S}$ on X_i, and we denote by δ_i their degrees. We use the notation (91), and by assumption we have (90). The local classification of rank 1 torsion-free sheaves $\mathcal{S}$ on X_t is simple. For $x \in \{x_1, x_2\}$, let $\mathcal{S}_x$ denote the stalk of $\mathcal{S}$ at x.

Lemma 9.1. *For any rank-1 torsion-free sheaf $\mathcal{S}$ over X_t and $x \in \{x_1, x_2\}$*

(1) either $\mathcal{S}_x$ is a rank-1 locally free $\mathcal{O}_{X_t,x}$-module,

(2) or $\mathcal{S}_x = \tilde{p}_* \tilde{\mathcal{S}}_x$ for some rank-1 locally free $\mathcal{O}_{\tilde{X}_b,x}$-module $\tilde{\mathcal{S}}_x$.

Proof. Similar to Lemma 8.4. □

The notion of bidegree is defined exactly as in 86. The global classification starts similarly as in Section 8.

Lemma 9.2. *For any fixed $(\delta_+, \delta_-) \in \mathbb{Z}^2$, there exists a unique isomorphism class of invertible sheaves $\tilde{\mathcal{S}}_{(\delta_+,\delta_-)}$ on $\tilde{X}_t$ of bidegree (δ_+, δ_-).*

Proof. Similar to Lemma 8.5. □

However, there is a small difference concerning invertible sheaves.

Lemma 9.3. *For any fixed $(\delta_+, \delta_-) \in \mathbb{Z}^2$, isomorphism classes of invertible sheaves $\mathcal{S}$ on X_t such that*

$$(\delta_+(\mathcal{S}), \delta_-(\mathcal{S})) = (\delta_+, \delta_-)$$

are parameterized by $\mathbb{C}^\times$.

Proof. It follows from the cohomology long exact sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}^\times_{X_t,x} \to \tilde{p}_*(\mathcal{O}^\times_{\tilde{X}_t})_x \to \mathbb{C}^\times \to 0.$$

□

This family of sheaves will be denoted by $\mathcal{S}_{(\delta_+,\delta_-)}(\lambda)$ with $\lambda \in \mathbb{C}^\times$. Following [18], we let

$$J(\mathcal{S}) = \{x \in \{x_1, x_2\} \mid \quad \mathcal{S}_x \text{ is locally an invertible sheaf}\}.$$

The invariant $2 - |J(\mathcal{S})|$ plays much the same role as $l(\mathcal{S})$ in Section 8, as they both measure the defect of $\mathcal{S}$ from being an invertible sheaf. In particular, we have

$$d = \delta_+ + \delta_- - 2 + |J(\mathcal{S})|,$$

and consequently (90) reads as

$$0 = (\delta_+ + \alpha_+) + (\delta_- + \alpha_-) - |J(\mathcal{S})|. \tag{109}$$

For any $i \in \{\pm\}$ restriction to the quotient sheaf

$$\mathcal{S} \to \mathcal{S}_{X_i} \to \mathcal{L}_i$$

gives rise to a quotient Higgs bundle $(\mathcal{Q}_i, \theta|_{\mathcal{Q}_i})$ of parabolic degree

$$\delta_i + \alpha_i.$$

Just as in Section 8, these are the only non-trivial quotient Higgs bundles of $(\mathcal{E}, \theta)$, so $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-semi-stable if and only if the two inequalities

$$\delta_i + \alpha_i \geq 0 \tag{110}$$

for $i \in \{\pm\}$ hold, and the condition for stability is given by the corresponding strict inequalities. Now, a straightforward modification of Lemma 8.12 shows that Hecke transformations allow us to reduce the other degree conditions to the case $d = -1$, i.e.

$$\alpha_+ + \alpha_- = 1.$$

We will distinguish the cases

$$\alpha_i \in (0, 1)$$

(case of generic weights) and

$$\alpha_i \in \{0, 1\}$$

(special weights).

9.1. **Degree -1, generic weights.** In this case, semi-stability is equivalent to stability. If $|J(\mathcal{S})| = 2$ then stability implies either

$$\delta_+ = 1, \delta_- = 0$$

or

$$\delta_+ = 0, \delta_- = 1,$$

so by virtue of Lemma 9.3 stable invertible sheaves are parameterized by

$$\mathbb{C}^\times_{(0,1)} \coprod \mathbb{C}^\times_{(1,0)}.$$

If $J(\mathcal{S}) = 1$, then $\mathcal{S}_x$ is locally free for exactly one of $x \in \{x_1, x_2\}$. It is easy to see that stability is equivalent to

$$\delta_+ = 0 = \delta_-,$$

so such stable sheaves are parameterized by a point for each of $x \in \{x_1, x_2\}$. We infer that

$$\begin{aligned}[\mathcal{M}^s_t(\vec{\alpha})] &= [\mathcal{M}^{ss}_t(\vec{\alpha})] \\ &= 2[\mathbb{C}^\times] + 2 \cdot \mathbf{1} \\ &= 2\mathbf{L}.\end{aligned}$$

As the moduli space is a smooth elliptic surface, it follows from Kodaira's list that the singular Hitchin fiber is of type I_2. This combined with Lemmas 7.1 and 7.2 finishes the proof of parts (5) and (6) of Theorem 2.1 and parts (2) and (3) of Theorem 2.4 in the case of generic $\vec{\alpha} \in W_1$.

Just as in Subsection 8.6, dropping the assumptions $\alpha_i^j \in [0,1)$ any value $\alpha_+ \in \mathbb{R}$ determines α_- by $\alpha_+ + \alpha_- = 1$, and it is a generic weight (in the sense that $\vec{\alpha}$-semistability is equivalent to $\vec{\alpha}$-stability) if and only if $\alpha_+ \notin \mathbb{Z}$. The stable bidegrees are again given by (102), while the corresponding Higgs bundles are now parameterized by

$$\mathbb{C}^\times_{(-\lceil\alpha_+\rceil+1, \lceil\alpha_+\rceil)} \coprod \mathbb{C}^\times_{(-\lceil\alpha_+\rceil+2, \lceil\alpha_+\rceil-1)}. \tag{111}$$

This shows the assertion of Theorem 1.3 in the particular case of a fiber X_t of type I_2, and provides the proof for the cases (5) and (6) of Theorem 2.1, and of cases (2) and (3) of Theorem 2.4.

9.2. **Degree -1, special weights.** We need to distinguish between stability and semi-stability. For sake of concreteness, we assume $\alpha_+ = 1, \alpha_- = 0$. If $|J(\mathcal{S})| = 2$, then stability implies

$$\delta_+ = 0, \delta_- = 1;$$

on the other hand, semi-stability allows for bidegrees

$$(-1, 2), (0, 1), (1, 0).$$

Let us denote by

$$(\mathcal{E}_{(1,0)}(\lambda), \theta_{(1,0)}(\lambda)) \tag{112}$$

the family of strictly semi-stable Higgs bundles corresponding to the family $\mathcal{S}_{(1,0)}(\lambda)$. Moreover, the family of sheaves $\mathcal{S}_{(-1,2)}(\lambda)$ also induces strictly semi-stable Higgs bundles denoted

$$(\mathcal{E}_{(-1,2)}(\mu), \theta_{(-1,2)}(\mu)) \tag{113}$$

for $\mu \in \mathbb{C}^\times$. Finally, let us denote by

$$(\tilde{\mathcal{E}}_{(\delta_+,\delta_-)}, \tilde{\theta}_{(\delta_+,\delta_-)}) \tag{114}$$

the Higgs bundle associated to $\tilde{p}_* \tilde{\mathcal{S}}_{(\delta_+,\delta_-)}$.

Lemma 9.4. *For any $\lambda, \mu \in \mathbb{C}^\times$ the Higgs bundles* (112) *and* (113) *are S-equivalent to* $(\tilde{\mathcal{E}}_{(-1,0)}, \tilde{\theta}_{(-1,0)})$.

Proof. Similar to Lemma 8.15. □

If $|J(\mathcal{S})| = 1$, then again $\mathcal{S}_x$ is locally free for exactly one of $x \in \{x_1, x_2\}$, and semi-stability is equivalent to

$$\delta_+ = 0 = \delta_-.$$

This produces two further strictly semi-stable Higgs bundles

$$(\mathcal{E}^x_{(0,0)}, \theta^x_{(0,0)}) \tag{115}$$

with $J(\mathcal{S}) = \{x\}$.

Lemma 9.5. *For both $x \in \{x_1, x_2\}$ the Higgs bundles* (115) *are S-equivalent to* $(\tilde{\mathcal{E}}_{(-1,0)}, \tilde{\theta}_{(-1,0)})$.

Proof. Similar to Lemma 8.16. □

From these lemmas we deduce that

$$[\mathcal{M}^s_t(\vec{\alpha})] = \mathbf{L} - \mathbf{1}$$
$$[\mathcal{M}^{ss}_t(\vec{\alpha})] = \mathbf{L}.$$

This combined with Lemmas 7.1 and 7.2 finishes the proof of parts (5) and (6) of Theorem 2.1 and parts (2) and (3) of Theorem 2.4 in the case of special weights.

10. Sheaves on singular fibers in the degenerate cases

In this section we will prove Theorems 2.5 and 2.7: we assume that the Higgs field is required to have a first order pole with non-semisimple residue at the marked point q_2, and study sheaves on the singular fibers of the fibration described in Propositions 6.2 and 6.4 that give rise to Higgs fields of the required local form. By virtue of Propositions 6.2 and 6.4, in this situation independently of the values of the parameters A, B, L, M, Q, R the elliptic fibration has a singular fiber X_t of type I_2, I_3, III or IV. In the above singular cases (as well as in the smooth case) one needs a thorough understanding of the relationship between torsion-free rank-1 sheaves on the singular curves in the pencil and singularity behaviour of Higgs bundles; this will be the content of Lemma 10.1. Indeed, due to the possible degeneration of nilpotent endomorphisms into semi-simple ones, a compactification phenomenon appears that has not yet been observed previously. Roughly speaking, Lemma 10.1 tells us that in order to find the Higgs bundles with non-semisimple residue we need to extract the invertible sheaves from the families found in Sections 7, 8 and 9. The torsion-free sheaves on the spectral curve that are not invertible give rise to Higgs bundles with diagonal residue at q_2. Again, Hecke transformations allow one to reduce any degree condition to degree -1, so we will only be interested in this most general case.

One needs to consider sheaves $\mathcal{S}$ on $\mathbb{F}_2$ whose support is a ramified double cover of $\mathbb{C}P^1$ by the map $p \circ \sigma$. Now, it follows from the results of Lemma 4.5 that in the cases considered here one of the components of X_t is the exceptional divisor E of the first blow-up at the base point P given by

$$P = (z_2 = 0, w_2 = b_{-1})$$

corresponding to the non-semisimple residue at the point q_2, see the dashed curve in Figure 8. In particular, the exceptional divisor E_1 then belongs to the fiber $(p \circ \sigma)^{-1}(q_2)$. Remember that we denote by

$$Z_t = \omega(X_t)$$

the singular curve in the pencil whose proper transform is X_t:

- if X_t is of type I_2 then Z_t is a nodal rational curve with a single node on the fiber of multiplicity 1;
- if X_t is of type I_3 then Z_t is composed of two rational curves intersecting each other transversely in two distinct points, one of them being on the fiber of multiplicity 1;
- if X_t is of type III then Z_t is a cuspidal rational curve with a single cusp on the fiber of multiplicity 1;

- if X_t is of type IV then Z_t is composed of two rational curves tangent to each other to order 2 on the fiber of multiplicity 1.

Let $(\mathcal{E},\theta)$ be an irregular Higgs bundle such that

$$Z_t = (\det(\zeta - p^*\theta)).$$

Let us denote by $\mathcal{S}$ the spectral sheaf of $(\mathcal{E},\theta)$:

$$0 \to p^*\mathcal{E} \otimes K^\vee(-3\{q_1\} - \{q_2\}) \xrightarrow{\zeta - p^*\theta} p^*\mathcal{E} \to \mathcal{S} \to 0. \tag{116}$$

By assumption, $\mathcal{S}$ is supported on Z_t. We will use the notations of Subsection 6.1. In particular, near q_2 we have $\zeta = w_2\kappa_2$ and $\theta = \vartheta_2\kappa_2$.

Lemma 10.1. *(1) If $\mathrm{Res}_{q_2}\theta$ is in the adjoint orbit* (54) *and Z_t is smooth then Z_t is ramified over q_2.*

(2) For any curve Z_t in the corresponding pencil (smooth or singular), the endomorphism $\mathrm{Res}_{q_2}\theta$ is in the adjoint orbit (54) *if and only if $\mathcal{S}$ is a locally free sheaf on Z_t near P.*

Proof. Up to a transformation $\zeta \mapsto \zeta + b_1$ we may assume $b_1 = 0$. Let us denote by $\mathfrak{m} = (z_2)$ the maximal ideal of $\mathbb{C}P^1$ at q_2.

For the first statement, with respect to the trivialization κ_2 we write

$$\vartheta_2 = \begin{pmatrix} a(z_2) & b(z_2) \\ c(z_2) & d(z_2) \end{pmatrix}.$$

Up to a suitable constant change of basis, our assumption on the adjoint orbit means that

$$b(0) = 1, \quad a(0) = c(0) = d(0) = 0. \tag{117}$$

We then have

$$f_2(z_2) = -(a(z_2) + d(z_2)), \quad g_2(z_2) = (a(z_2)d(z_2) - b(z_2)c(z_2)).$$

Because of (117) we have $f_2(0) = 0$ and

$$a(z_2)d(z_2) - b(z_2)c(z_2) \equiv -c(z_2) \pmod{\mathfrak{m}^2}. \tag{118}$$

It follows that the characteristic polynomial $\chi_{\vartheta_2}(z_2)$ is an Eisenstein polynomial if and only if

$$c(z_2) \not\equiv 0 \pmod{\mathfrak{m}^2}.$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial \chi_{\vartheta_2}}{\partial w_2} &= 2w_2 - (a(z_2) + d(z_2)) \\
\frac{\partial \chi_{\vartheta_2}}{\partial z_2} &= -\frac{\mathrm{d}(a(z_2) + d(z_2))}{\mathrm{d}z_2} w_2 + \frac{\mathrm{d}(a(z_2)d(z_2) - b(z_2)c(z_2))}{\mathrm{d}z_2}.
\end{aligned}$$

Plugging $z_2 = 0 = w_2$ into these expressions and using (117) we get

$$\begin{aligned}
\frac{\partial \chi_{\vartheta_2}}{\partial w_2}(0,0) &= 0 \\
\frac{\partial \chi_{\vartheta_2}}{\partial z_2}(0,0) &= \frac{\mathrm{d}(a(z_2)d(z_2) - b(z_2)c(z_2))}{\mathrm{d}z_2}.
\end{aligned}$$

Now, because of (118) the curve defined by χ_{ϑ_2} is singular at $(0,0)$ if and only if

$$c(z_2) \equiv 0 \pmod{\mathfrak{m}^2}.$$

To sum up, if the curve defined by χ_{ϑ_2} is smooth then χ_{ϑ_2} is an Eisenstein polynomial, hence ramified over 0. As for the second statement, tensoring the identity (116) with

$$p^*(\mathcal{O}_{\mathbb{C}P^1}/\mathfrak{m}) = \mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m} \tag{119}$$

over $\mathcal{O}_{\mathbb{F}_2}$ yields

$$\cdots \to \mathcal{T}or_1^{\mathcal{O}_{\mathbb{F}_2}}(\mathcal{S}, \mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m}) \to$$
$$\to p^*(\mathcal{E} \otimes K^\vee(-3\{q_1\} - \{q_2\})/\mathfrak{m}) \xrightarrow{\zeta - p^*\theta \pmod{\mathfrak{m}}} p^*(\mathcal{E}/\mathfrak{m}) \to \mathcal{S} \otimes \mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m} \to 0,$$

where $\pmod{\mathfrak{m}}$ stands for the morphism induced on the reduction modulo $\mathfrak{m}$. By the symmetry of the $\mathcal{T}or$ functor we have

$$\mathcal{T}or_1^{\mathcal{O}_{\mathbb{F}_2}}(\mathcal{S}, \mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m}) = \mathcal{T}or_1^{\mathcal{O}_{\mathbb{F}_2}}(\mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m}, \mathcal{S}) = 0$$

because $\mathcal{S}$ is a torsion-free $\mathcal{O}_{\mathbb{F}_2}$-module. We infer that

$$\mathcal{S} \otimes \mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m} = \operatorname{coker}(\zeta - p^*\theta \pmod{\mathfrak{m}}).$$

Locally near $z_2 = 0$ the ring $\mathcal{O}_{\mathbb{F}_2}$ is given by $\mathbb{C}[z_2, w_2]$ and (119) is given by $\mathbb{C}[w_2]$. In different terms, we have the equality of $\mathbb{C}[w_2]$-modules

$$\mathcal{S} \otimes_{\mathbb{C}[z_2,w_2]} \mathbb{C}[w_2] = \operatorname{coker}(w_2 - \vartheta_2(0)). \tag{120}$$

This (and the assumption $b_1 = 0$) implies that the fiber of $\mathcal{S}$ vanishes over $(0, w_2)$ for any $w_2 \neq 0$. On the other hand, $\vartheta_2(0)$ can be identified with the residue of

$$\theta: \mathcal{E} \otimes K^\vee(-\{q_2\}) \to \mathcal{E}$$

at q_2. Reducing (120) modulo w_2 we find that the cokernel of $\operatorname{Res}_{q_2}(\theta)$ is of dimension 1 if and only if the fiber of $\mathcal{S}$ at $(0,0)$ is of dimension 1, i.e. if and only if $\mathcal{S}$ is locally free near $(0,0)$. □

Recall the definition of generic and special weights from Definition 8.14. We will use the same notion of genericity, up to the convention (5) in the case of a Higgs field with non-semisimple polar part or non-semisimple residue at q_j.

Lemma 10.2. *The Hitchin fibers of the moduli spaces $\mathcal{M}_t^s, \mathcal{M}_t^{ss}$ over t are given by*

(1) if X_t is of type I_2 then $[\mathcal{M}_t^s(\vec{\alpha})] = [\mathcal{M}_t^{ss}(\vec{\alpha})] = \mathbf{L} - \mathbf{1}$,

(2) if X_t is of type I_3 then
   - *(a) for generic weights we have $[\mathcal{M}_t^s(\vec{\alpha})] = 2\mathbf{L} - \mathbf{1}$,*
   - *(b) for special weights we have*

$$[\mathcal{M}_t^s(\vec{\alpha})] = \mathbf{L}$$
$$[\mathcal{M}_t^{ss}(\vec{\alpha})] = \mathbf{L} + \mathbf{1}.$$

(3) if X_t is of type III then $[\mathcal{M}_t^s(\vec{\alpha})] = \mathbf{L}$,

(4) if X_t is of type IV then
   - *(a) for generic weights we have $[\mathcal{M}_t^s(\vec{\alpha})] = 2\mathbf{L}$,*
   - *(b) for special weights we have*

$$[\mathcal{M}_t^s(\vec{\alpha})] = \mathbf{L}$$
$$[\mathcal{M}_t^{ss}(\vec{\alpha})] = \mathbf{L} + \mathbf{1}.$$

Proof. (1) If X_t is of type I_2 then Z_t is a nodal rational curve with a single node on the fiber of multiplicity 1. As Z_t has a unique component, stability is automatic, thus S-equivalence is the same relation as isomorphism. The degree of $\mathcal{S}$ must satisfy (85), and Lemma 10.1 implies that $\mathcal{S}$ must be a locally free sheaf on Z_t. The result follows from Lemma 7.1.

(2) If X_t is of type I_3 then Z_t is composed of two rational curves intersecting each other transversely in two distinct points, one of them being on the fiber of multiplicity 1. By Lemma 10.1, $\mathcal{S}$ must be locally free on the fiber over q_2, i.e. $J(\mathcal{S}) \supseteq \{q_2\}$. Equation (109) is valid, and the analysis closely follows the one detailed in Section 9, hence we content ourselves with sketching it. In the case of generic weights and everywhere locally free sheaves $J(\mathcal{S}) = \{q_1, q_2\}$, there exist

two bidegree conditions (102) compatible with stability, giving rise to a family of sheaves parameterized by two copies of $\mathbb{C}^{\times}$, specifically (111). In the case of generic weights and $J(\mathcal{S}) = \{q_2\}$, we get a further S-equivalence class of sheaves. This argument finishes the proof of Theorem 1.3 in the case corresponding to (3) of Theorem 2.5. If the weights are special, then stability is equivalent to a unique value of the bidegree, giving rise to a family of locally free sheaves parameterized by $\mathbb{C}^{\times}$, plus one further sheaf with $J(\mathcal{S}) = \{q_2\}$. On the other hand, semi-stability is equivalent to any one of three bidegree conditions: two extremal ones in addition to the central one given by stable sheaves. The semi-stable sheaves with one of the two extremal bidegrees are all S-equivalent to each other, so the family of semi-stable Higgs bundles coming from locally free sheaeves is parameterized by $\mathbb{C}$. Again, there is one further sheaf with $J(\mathcal{S}) = \{q_2\}$.

(3) If X_t is of type III then Z_t is a cuspidal rational curve with a single cusp on the fiber of multiplicity 1. Stability is again automatic, δ satisfies (85), and $\mathcal{S}$ must be a locally free sheaf on Z_t. We conclude using Lemma 7.2.

(4) If X_t is of type IV then Z_t is composed of two rational curves tangent to each other to order 2 on the fiber of multiplicity 1. This analysis closely follows the one carried out in Subsection 8.6, hence we only give a sketch. For generic weights, stability is equivalent to (102) and Lemma 8.6 shows that stable Higgs bundles are parameterized by (103). This verifies Theorem 1.3 in case (1) of Theorem 2.5. In the case of special weights, the $l(\mathcal{S}) = 0$ part of the analysis of Subsection 8.7 can be repeated verbatim. Namely, there exists a unique bidegree condition compatible with stability, while there are three bidegree conditions compatible with semi-stability. Thus, the stable Higgs bundles are parameterized by $\mathbb{C}$. The semi-stable ones having one of the two extremal bidegrees are all S-equivalent to each other (and actually, to all semi-stable ones with $l(\mathcal{S}) \in \{1, 2\}$), so they give rise to a point in the corresponding Hitchin fiber.

□

The lemma combined with Lemmas 7.1 and 7.2 finishes the proof of Theorems 2.5, 2.7 and 1.3.

REFERENCES

[1] A. Altman and S. Kleiman. The presentation functor and the compactified Jacobian. In *The Grothendieck Festschrift*, volume 86 of *Progress in Mathematics*, pages 15–32. Birkhauser, 1990.

[2] O. Biquard and Ph. Boalch. Wild non-abelian Hodge theory on curves. *Compos. Math.*, 140(1):179–204, 2004.

[3] P. Cook. *Local and Global aspects of the Module Theory of Singular Curves*. PhD thesis, University of Liverpool, 1993.

[4] D. Gaiotto, G. Moore, and A. Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013.

[5] R. Donagi and T. Pantev. Langlands duality for Hitchin systems. *Invent. Math.*, 189(3):653–735, 2012.

[6] P. Gothen and A. Oliveira. The singular fiber of the Hitchin map. *Int. Math. Res. Not. IMRN*, 5:1079–1121, 2013.

[7] G.-M. Greuel and H. Knörrer. Einfache Kurvensingularitäten und torsionsfreie Moduln. *Math. Ann.*, 270(3):417–425, 1985.

[8] J. Harer, A. Kas, and R. Kirby. Handlebody decompositions of complex surfaces. *Mem. Amer. Math. Soc.*, 62(350):iv+102, 1986.

[9] T. Hausel, A. Mellit, and D. Pei. Mirror symmetry with branes by equivariant verlinde formulae. arXiv:1712.04408.

[10] T. Hausel and M. Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.

[11] N. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[12] N. Hitchin. Higgs bundles and characteristic classes. In *Arbeitstagung Bonn, In memory of Friedrich Hirzebruch*, Progress in Mathematics, pages 247–264. Birkhauser, 2016.

[13] P. Ivanics, A. Andras Stipsicz, and Sz. Szabó. Hitchin fibrations on two-dimensional moduli spaces of irregular higgs bundles with one singular fiber. arXiv:1808.10125.

[14] P. Ivanics, A. Stipsicz, and Sz. Szabó. Two-dimensional moduli spaces of rank 2 Higgs bundles over $\mathbb{C}P^1$ with one irregular singular point. *J. Geom. Phys.*, 130:184–212, 2018.
[15] K. Kodaira. On compact analytic surfaces: II. *Ann. Math.*, 77:563–626, 1963.
[16] R. Miranda. Persson's list of singular fibers for a rational elliptic surface. *Math. Z.*, 205(2):191–211, 1990.
[17] T. Mochizuki. *Wild harmonic bundles and wild pure twistor D-modules*. Number 340 in Astérisque. Société Mathématique de France, 2011.
[18] T. Oda and C. Seshadri. Compactifications of the generalized Jacobian variety. *Transactions of the AMS*, 253, 1979.
[19] U. Persson. Configurations of Kodaira fibers on rational elliptic surfaces. *Math. Z.*, 205(1):1–47, 1990.
[20] C. Sabbah. Harmonic metrics and connections with irregular singularities. *Annales Institut Fourier*, 49, 04 1999.
[21] C. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Bull. AMS*, 83(1), 1976.
[22] A. Stipsicz, Z. Szabó, and Á. Szilárd. Singular fibers in elliptic fibrations on the rational elliptic surface. *Periodica Mathematica Hungarica*, 54:137–162, 2007.
[23] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nuclear Phys. B*, 479(1-2):243–259, 1996.
[24] Sz. Szabó. Perversity equals weight for painlev systems. arXiv:1802.03798.
[25] Sz. Szabó. The birational geometry of unramified irregular Higgs bundles on curves. *Intern. J. Math.*, 28(6), 2017.

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, 1111. BUDAPEST, EGRY JÓZSEF UTCA 1. H ÉPÜLET, HUNGARY
E-mail address: `ipe@math.bme.hu`

RÉNYI INSTITUTE OF MATHEMATICS, 1053. BUDAPEST, REÁLTANODA UTCA 13-15. HUNGARY
E-mail address: `stipsicz@renyi.hu`

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, 1111. BUDAPEST, EGRY JÓZSEF UTCA 1. H ÉPÜLET, HUNGARY
E-mail address: `szabosz@math.bme.hu`